R Buchanan

1830

RECREATIONS

IN

MATHEMATICS

AND

NATURAL PHILOSOPHY.

IN FOUR VOLUMES.

VOL. I.

RECREATIONS

IN

MATHEMATICS

AND

NATURAL PHILOSOPHY:

CONTAINING

AMUSING DISSERTATIONS AND ENQUIRIES CONCERNING A VARIETY OF SUBJECTS THE MOST REMARKABLE AND PROPER TO EXCITE CURIOSITY AND ATTENTION TO THE WHOLE RANGE OF THE MATHEMATICAL AND PHILOSOPHICAL SCIENCES:

The Whole treated in a pleasing and easy Manner, and adapted to the Comprehension of all who are the least initiated in those Sciences: viz.

ARITHMETIC,
GEOMETRY,
TRIGONOMETRY,
MECHANICS,
OPTICS,
ACOUSTICS,
MUSIC,

ASTRONOMY,
GEOGRAPHY,
CHRONOLOGY,
DIALLING,
NAVIGATION,
ARCHITECTURE,
PYROTECHNY,

PNEUMATICS,
HYDROSTATICS,
HYDRAULICS,
MAGNETISM,
ELECTRICITY,
CHEMISTRY,
PALINGENESY, &c.

FIRST COMPOSED BY

M. OZANAM, OF THE ROYAL ACADEMY OF SCIENCES, &c.

LATELY RECOMPOSED, AND GREATLY ENLARGED, IN A NEW EDITION, BY THE CELEBRATED

M. MONTUCLA.

AND NOW TRANSLATED INTO ENGLISH, AND IMPROVED WITH MANY ADDITIONS AND OBSERVATIONS, BY

CHARLES HUTTON, LL.D. AND F.R.S.

EMERITUS PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH.

IN FOUR VOLUMES;

WITH NEAR ONE HUNDRED COPPER-PLATES.

VOL. I.

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1814.

ADVERTISEMENT

BY THE EDITOR.

THE Recreations, Mathematical and Philosophical, of M. Ozanam, have always been justly esteemed the most rational and complete of all that have ever been composed in any country; having gone through numerous editions, and been translated into various languages, even in their rudest state and form.

Many things, however, in the original work required much alteration and improvement, in corrections, alterations, and additions, to adapt it to the actual improved state of the liberal sciences. Accordingly, this task has been very ably performed by M. Montucla, the very learned author of the celebrated History of the Mathematics, who has just published a new edition of those *Recreations*, in four large volumes, embellished with a great number of elegant copper plates. Under his hand the work assumes a quite different form and appearance, having been wholly new cast and composed, the puerilities and improprieties expunged, the materials properly arranged, and the whole greatly enlarged, with the new sciences and the numerous improvements that have been made for nearly the last 200 years, since Ozanam first compiled his original work. So that the whole appears now rather as a new work, of the present times, than a new edition of the old one. The circumstances of which are particularly described in M. Montucla's own preface.

The excellence of Montucla's work, then, has induced me to render it still more useful to the English reader, by the present translation; which is also further augmented by the addition of many notes, remarks, and dissertations, relating to several particulars which have been omitted even by Montucla himself; which the reader will find dispersed through all the volumes and all the parts of this work; a work in which will be found an easy and familiar account of every thing the most amusing and curious in all the branches of the mathematical and philosophical sciences: Thus, in Arithmetic; we have the different systems and kinds of arithmetic; short and curious ways of computing; arithmetical machines and Napier's rods; palpable arithmetic; curious properties of numbers, perfect, amicable, prime, squares, cubes, &c, figurate, triangular, &c, pyramidal, progressions, musical; combinations, probabilities, chances, sports and pastimes, divinations or guesses, cards, dice, magic squares and circles; Political arithmetic, proportion of the males to the females, the numbers of persons of all ages, proportion of births to the number of persons and families, &c, &c, &c.—In Geometry, the various properties, constructions, transformations, and measures of geometrical figures; as triangles, squares, parallelograms, trapeziums, polygons, circles, lunes, ellipses, spheres, &c; quadrature and rectification of the circle; geometrical problems, both on paper and on the ground; select and new geometrical theorems, more extensive and general than formerly; the most advantageous form and position of the cells in honey comb, &c.—In Mechanics, curious and interesting problems, properties and machines; history of the attempts at the perpetual motion, and of celebrated machines, both ancient and modern; water-wheels, steam-engines, &c.—In Optics, the new and important discoveries, both microscopical and others.—In Acoustics, Pneumatics, and Music, the formation and propagation of sounds; echoes, pneumatic engines, ancient and modern music, effects of harmony, description of musical instruments, &c.—In Astronomy and Geography, finding meridian lines, latitude, longitude, time, climates, measures of degrees, figure and magnitude of the earth, maps, the stars, planets, sun, moon, comets, eclipses, constellations, system of the universe, chronology, calendars, epochs, &c.—On Gnomonics or Dialling, all kinds of curious dials, both on plane and curved surfaces.—In Navigation, the governing and manœuvring of ships, finding the latitude and longitude at sea, history of the longitude, &c.—In Architecture, the construction of walls, vaults, arches, bridges, domes, &c. —In Pyrotechny, the mixture of powder and compositions, for muskets, cannon, all kinds of fire-works, stars, rockets, serpents, marroons, jets, wheels, suns, fire that burns under water, &c.—In Chemistry, Philosophy, &c. of fire, air, water, earth, thunder, winds, hydraulics, hydrostatics, barometers, thermometers, hygrometers, air-pumps, water-pumps, syphons, fountains, odours, light, heat, cold, ice, magnets natural and artificial, electric fire, lightening, metals, earths, salts, phosphorus, sympathetic inks, metallic vegetations, perpetual lamps, palingenesy, &c, &c. The particulars of all which are stated at length in the table of contents at the beginning of each volume; besides the more ample account of the whole work, in the preface inserted in the first part, by Montucla, the learned editor of the new French edition.

Royal Mil. Acad.
Woolwich, Oct. 20, 1801. }

CHA. HUTTON.

SOME ACCOUNT

OF THE

LIFE AND WRITINGS

OF

MONTUCLA.

JOHN STEPHEN MONTUCLA, member of the National Institute, and of the academy of Berlin, censor royal for mathematical books, and author of this new-modelled and enlarged edition of the Mathematical Recreations of Ozanam, was born at Lyons, the 5th of September 1725. His father was a banker, by whom he was intended for the same profession; but the science of calculations, to which he was early introduced, soon produced a discovery of the natural bent of his mind. In the Jesuits college at Lyons he laid a good foundation in the ancient languages, as well as in the mathematical sciences, which enabled him afterwards easily to acquire a competent acquaintance with the Italian, the German, the Dutch, and the English, which he not only read, but also spoke very well.

At 16 years of age Montucla lost his father; and his grandmother, who had been left guardian of his education, died 4 years after. Having finished his studies at Lyons, he went to Toulouse to study the law, a branch of study deemed necessary in the liberal education of every person not destined for the profession of arms.

From hence he repaired to Paris, to enjoy in that capital all the benefits it afforded to the studious, in the lessons of the best masters, in the rich collections of the productions of nature and art, in the best libraries of books, and in the united societies of the literati, such as Diderot, Dalembert, Degua, Lalande, Blondel, Cochin, Courtou, le Blond, &c, among whom he found friends for the rest of his life, and which fixed and determined his choice and pursuit of the mathematical and philosophical sciences, in which he afterwards distinguished himself in so eminent a degree. It was only in relaxing and unbending his mind, from such severe exercises, that he could sometimes occupy himself privately on subjects of less magnitude: such as when he in a manner made an entire new book of *Ozanam's Mathematical Recreations*, by the multitude of articles added, abridged, or substituted: on which occasion he had so closely concealed from every person the secret of his concern in that neat and improved edition, that the work was actually sent to him to examine and authorize in his capacity of public censor for mathematical books, an honorary office to which he had some time before been appointed. To the last edition of these Recreations however, from whence these four volumes have been translated, he set the initials of his name.

Many other pieces were in the like anonymous manner composed by Montucla; among which may be here noticed an ingenious and learned *History of Researches relating to the Quadrature of the Circle*, published in 1754; a work very interesting, on account of the number of speculators who have gone astray after that seducing phantom, and of the curious properties which the researches have given rise to.

On occasion of introducing into France, in 1756, the practice of inoculation, which had been brought to England in 1721, by lady Montague, on her return from Constantinople, Montucla made a translation from the English of the principal writings on that subject, which he added to the Memoire of la Condamine.

In the year 1758, came out Montucla's grand work, the *History of Mathematics*, in 2 large volumes in quarto: a work of profound reading and learning, and upon which, young as he was, he had spent a great many years of his life. This performance, of immense labour and erudition, published at 33 years of his age, justly procured to the author a most distinguished place in the learned world. This history, so truly admirable, whether we consider the extreme clearness and precision with which the subjects are treated, or the profound learning it exhibits, having been long out of print, the author's employment under the government, as first commissary of the king's buildings, for many years prevented him from fully yielding to the solicitations of his learned friends, to continue the work through the 18th century, in a new and enlarged edition. But the unfortunate loss of his fortune and employment, by the late revolution in France, left him but too much leisure for that purpose. The consequence, happy in this instance for the sciences, has been a new edition in 4 large volumes; in which the history is continued down to the end of the 18th century, and the former parts also very much enlarged and corrected.

In 1755, Montucla was elected an associated member of the academy at Berlin. And in 1761 he was placed at Grenoble as secretary to the office of intendance, where he united in a happy marriage with Maria Françoise Romand, who was still living at his death.

The duke de Choiseul having ordered, in 1764, a colony to be formed at Cayenne, Montucla went out there as first secretary to the commission, to which appointment was joined also that of astronomer royal. The affairs of the colony not proving successful, after 15 months Montucla returned again to Grenoble, bringing with him many useful observations and specimens in botany and natural history, which proved beneficial both to the sciences and to the public at large. This voyage also furnished him with those curious observations on the shining of the sea in many places, and of various luminous insects, which are inserted near the end of the 4th volume of these Recreations.

Soon after his return, Montucla was appointed at Versailles to the honourable and profitable office of first commissioner of the royal and public buildings; an employment which he executed with great ability and usefulness during more than 25 years, till the overthrow of the monarchy put an end at once to his office, and the little fortune his regularity and œconomy had enabled him to save, throwing him again on the world, in his old age, naked and stript of every thing except his integrity, and the love and respect of his friends!

The modesty and integrity of Montucla were not less remarkable than his erudition. He was offered a place in the Academy of Sciences of Paris; which through delicacy he refused, as he felt he should not have leisure sufficient properly to attend to the duties of it. The portions of time which others would give to their pleasures, or amusements in their families, he always devoted to the details of the duties of his office, or to his studies. The translation from the English, of *Carver's Travels in North America*, was the sole monument of his pen, during that long interval. And even this was produced properly in the faithful discharge of the public duties with which he was charged. Being particularly entrusted by the government with the correspondence relating to the voyages which it ordered, he made it his duty and care to collect all the accounts he could find relating to such enterprizes by other countries. With this view, at first only amusing his family with the reading of Carver's travels, finding it entertaining and instructive, he completed and published the whole translation.

Montucla was named a member of the National Institute from the time of its commencement. And the government of 1795 employed him in examining and analyzing the treatises deposited in the national archives. He was named professor of mathematics of the central school at Paris; but the bad state of his health would not permit him to accept it; and the Department honoured him with a place in the jury of central instruction. But a place in the office for the national lottery was the only resource for his family during two years; a pension of 2400 francs (100 l.), given him by the minister Neufchateau on the death of Saussure, and which he enjoyed only four months before his decease, which happened the 18th of December 1799. It was chiefly occasioned, as it often happens to literary and sedentary men, by a retention of urine: leaving a widow, as also a daughter, married in 1783, and a son employed in the office of the minister for the interior.

Montucla was one of the many considerable mathematicians of the 18th century; being well acquainted with all the branches and improvements in those abstruse sciences. His taste however, always chaste and clear, led him to prefer the pure and luminous methods of the ancient mathematicians, and to blame, in the French and the Germans, the great neglect of the same principles, which they showed on all occasions by their preference of the mere modern analysis.

In the qualities of his heart too Montucla was truly estimable: remarkably modest in his manner and deportment; benevolent far beyond the means of his small fortune: of a very respectable personal appearance; he spoke with ease and precision, but unassuming and with simplicity; related anecdotes and stories in a pleasant and playful manner; and breathing, in all his conduct and deportment, the sweetness of virtue, and the delicacy of a fine taste.

ON THE

LIFE AND WRITINGS

OF

OZANAM,

THE FIRST AUTHOR OF THESE

MATHEMATICAL RECREATIONS.

JAMES OZANAM, whose fame is established as an eminent Mathematician, was born at Boligneux in Bressia, in the year 1640: he was descended from a family of Jewish extraction, but which had long been converts to the Romish faith, and some of whom had held considerable places in the parliaments of Provence. Being a younger son, though of an opulent family, it was thought proper to educate him for the church, that he might be qualified for some small benefices belonging to the family: he accordingly studied divinity four years, but this was purely in obedience to the will of his father, on whose death he relinquished his theological pursuits, and, following his natural inclinations, devoted himself to the study of the mathematics. Having considerable genius, as well as much industry, he made very great progress, though unassisted by a master, and at the juvenile age of 15 years he wrote a mathematical treatise.

While very young he removed to Lyons, and, for a maintenance, taught the mathematics, with tolerable success: but his generosity soon procured him a better residence. Among his pupils were two foreigners, who, being disappointed of some bills of exchange for a journey to Paris, mentioned the circumstance to him: finding that

50 pistoles were necessary to enable them to accomplish their purpose, he immediately supplied them with the money, even without their note for it. On their arrival at Paris, they mentioned this generous action to M. Dugusseau, father of the chancellor; who being struck with this trait in his character, engaged these young gentlemen to invite Ozanam to Paris, with a promise of his favour. He embraced this opportunity with eagerness, and, at Paris, the employment of giving instructions in mathematics soon brought him in a considerable income; though his business, however, procured him plenty of money, he *saved* none; for, being addicted both to gaming and gallantry, these continually drained his purse. After a few years of dissipation and expense he began to wish for domestic enjoyments, and soon entered into the connubial state with a young woman, who, though she brought him no fortune, was formed to give him happiness, being discreet, modest, virtuous, and of a sweet disposition. From the period of his marriage, he long enjoyed much comfort, and, besides attending to his business as a mathematical master, he wrote a great number of useful works. Among these we cannot help mentioning his Treatise on Lines of the first Order, and on the Construction of Equations, published in 1687: the Mathematical Dictionary, published in 1690: the Course of Mathematics, 5 volumes octavo, published in 1693: the Mathematical and Philosophical Recreations, first published in 1694, in 2 vols. 8vo: and the Elements of Algebra, in 2 vols. 8vo. published in 1702.

Our author had 12 children, but had the pain of losing most of them while young: and, to complete his bereavements, his wife died, in 1701, which last stroke made him truly unhappy. About this period too the war breaking out, on account of the Spanish succession, it deprived him of most of his pupils, who, being foreigners, were obliged to leave Paris. This concurrence of painful circumstances reduced him to a very melancholy state; from which he had merely a temporary relief, in consequence of his being admitted an eleve of the Royal Academy of Sciences; but he never recovered his wonted health and spirits; so that, though he lingered through a few dull years, with a strong presentiment of his approaching dissolution, he might rather be said to exist than to live, until the year 1717, when he was seized with an apoplexy, which terminated his existence on the third of April, at 77 years of age.

Ozanam possessed a mild and calm disposition, a cheerful and pleasant temper, an inventive genius, and a generosity almost unparalleled. After marriage his conduct was irreproachable: and, at the same time that he was sincerely pious, he had a great aversion to disputes about theology. On this subject he used to say, that it was the business of the Sorbonne doctors to discuss, of the Pope to decide, and of a *Mathematician to go straight to heaven in a perpendicular line.*

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PART II.

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NOTE. In binding the volumes, it is to be observed that the plates relating to each of the parts or sciences are to be placed all together at the end of that part. Thus, all the plates relating to the first part, or Arithmetic, to be placed in the middle of the first volume, at the end of the arithmetic; then all the plates relating to Geometry to be placed after that science at the end of the volume. And so of the others.

MATHEMATICAL

AND

PHILOSOPHICAL

RECREATIONS.

PART FIRST.

Containing the most curious Problems, and most interesting Truths, in regard to Arithmetic.

ARITHMETIC and geometry, according to Plato, are the two wings of the mathematician. The object indeed of all mathematical questions, is to determine the ratios of numbers, or of magnitudes; and it may even be said, to continue the comparison of the ancient philosopher, that arithmetic is the mathematician's right wing; for it is an incontestable truth, that geometrical determinations would, for the most part, present nothing satisfactory to the mind, if the ratios thus determined could not be reduced to numerical ratios. This justifies the common practice, which we shall here follow, of beginning with arithmetic.

This science affords a wide field for speculation and curious research; but in the collection which we here present to the reader, we have confined ourselves to what is best calculated to excite the curiosity of those who have a taste for mathematical pursuits.

CHAPTER I.

Of our Numerical System, and the different Kinds of Arithmetic.

IT has been generally remarked, that all, or most of the nations with which we are acquainted, reckon by periods of ten; that is to say, after having counted the units from 1 to 10, they begin and add units to the ten; having attained to two tens or 20, they continue to add units as far as 30, or three tens; and so on in succession, till they come to ten tens, or a hundred; of ten times a hundred they form a thousand, and so on. Did this arise from necessity; was it occasioned by any physical cause; or was it merely the effect of chance?

No person, after the least reflection on this unanimous agreement, will entertain any idea of its being the effect of chance. It is not only probable, but might almost be proved, that this system derives its origin from our physical conformation. All men have ten fingers; a very few excepted, who, by some lusus naturæ, have twelve. The first men began to reckon on their fingers. When they had exhausted them by reckoning the units, it was necessary that they should form a first total, and again begin to reckon the same fingers, till they exhausted them a second time; and so on in succession. Hence the origin of tens, which being confined, like the units, to the number of the fingers, could not be carried beyond it, without forming a new total, called a hundred; then another called a thousand, and so on.

From these observations, a curious consequence may be drawn. If nature, instead of ten fingers, had given us twelve, our system of numeration would have been different. After 10, instead of saying ten plus one or eleven, ten plus two or twelve, we should have ascended by simple denominations to twelve, and should then have counted twelve plus one, twelve plus two, &c, as far as two dozens; our hundred would have been twelve dozens, &c. A six-fingered people would certainly have had an arithmetic of this kind, which indeed would have sufficiently answered every arithmetical purpose, or rather would have been attended with various advantages, which our numerical system does not possess.

In consequence of an idea of this kind, philosophers have been induced to examine the properties of other numerical systems. The celebrated Leibnitz proposed one, in which only two characters, 1 and 0, were to be employed. In this system of arithmetic, the addition of an 0 multiplied every thing by two, as it does by ten in common arithmetic, and the numbers were expressed as follows:

One	1
Two	10
Three	11
Four	100
Five	101
Six	110
Seven	111
Eight	1000
Nine	1001
Ten	1010
Eleven	1011
Twelve	1100
Thirteen	1101
Fourteen	1110
Fifteen	1111
Sixteen	10000
Thirty-two	100000
Sixty-four	1000000
Two thousand three hundred and seventy-nine	100101001011

As Leibnitz found in the above mode of expressing numbers some peculiar advantages, he published, in the Memoirs of the Academy of Sciences at Berlin, rules for performing, in this kind of arithmetic, the usual operations of common arithmetic. But it may be readily perceived, that this new system, if introduced into practice, would be attended with the inconvenience of requiring too many figures: twenty would be necessary to express a number equal to about a million.

One curious circumstance, in regard to this binary arithmetic, must not be here omitted. It serves to explain, as some pretend, a Chinese symbol, which has occasioned great embarrassment to the learned who have applied to the study of the Chinese antiquities. This symbol, which is highly revered by the Chinese, who ascribe it to their ancient emperor Fohi, consists of certain characters formed by the different combinations of a small whole line and a broken one. Father Bouset, a celebrated Jesuit, who resided some time in China as a missionary, having heard of Leibnitz's ideas, observed, that if the whole line were supposed to represent our 1, and the broken line our 0, these characters would be nothing else than a series of numbers expressed by binary arithmetic. It is very singular, that a Chinese enigma should find its Œdipus only in Europe; but perhaps in this explanation there is more of ingenuity than truth.

If the binary arithmetic of Leibnitz is entitled to no farther notice, than to be classed among the curious arithmetical speculations, the case however is not the same with duodenary arithmetic, or that kind which, as already observed, would have been brought into use had men been born with twelve fingers. This arithmetic would indeed have been as expeditious as the arithmetic now employed, and even somewhat more so; the number of the characters, which would have received an increase only of two, to express ten and eleven, would have been as little burthensome to the memory as the present characters, and would have been attended with advantages which ought to make us regret that this system was not orginally adopted.

It is not improbable, however, that the duodenary system would have been preferred had philosophy presided at the invention; for it would have been readily seen that *twelve*, of all the numbers from 1 to 20, is that which possesses the advantage of being small, and of having the greatest number of divisors; for there are no less than four divisors by which it can be divided without a fraction, viz. 2, 3, 4 and 6. The number 18 indeed has four divisors also; but being larger than 12, the latter deserves to be preferred for measuring the periods of numeration. The first of these periods, from one to twelve, would have had the advantage of being divisibleby 2, 3, 4, 6; and the second, from one to 144, by 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72; whereas, in our system the first period, from one to 10, has only two divisors, 2 and 5; and the second, from one to a hundred, has only 2, 4, 5, 10, 20, 25, 50. It is evident therefore, that fractions would less frequently have occurred in the designation of numbers in that way, namely by twelves.

But what would have been most convenient in this mode of numeration, is that, in the divisions and sub-divisions of measures, it would have introduced a duodecimal progression. Thus, as the foot has by chance been divided into 12 inches, the inch into 12 lines, and the line into 12 points; the pound would have been divided into 12 ounces, the ounce into 12 drams, and the dram into 12 grains, or parts of any other denominations; the day would have been divided into 12 equal portions called hours, the hour into 12 other parts, each equal to 10 minutes, each of these parts into 12 others; and so on successively. The case would have been the same in regard to measures of capacity.

Should it be asked, what would be the advantages of such a division? we might reply as follows. It is well known by daily experience, that when it is necessary to divide any measure into 3, 4, or 6 parts, an integer number in the measures of a lower denomination cannot be found, or at least only by chance. Thus, the third or the 6th of a pound averdupois does not give an exact number of ounces; and the third of a pound sterling, does not give an integer number of shillings. The case is the same in regard to the bushel, and the greater part of the other measures of capacity. These inconveniences, which render calculations exceedingly complex, would not take place if the duodecimal progression were every-where followed.

There is still another advantage which would result from a combination of duodenary arithmetic, with this duodecimal progression. Any number of pounds, shillings, and pence; of feet, inches, and lines, or of pounds, ounces, &c, being given, they would be expressed as whole numbers of the same kind usually are in common arithmetic. Thus, for example, supposing the fathom to consist of 12 feet, as must necessarily be the case in this system of numeration, if we had to express 9 fathoms 5 feet, 3 inches and 8 lines, we should have no occasion to write $9^f\ 5^f\ 3^i\ 8^l$, but merely 9538; and whenever we had a similar number expressing any dimension in fathoms, feet, inches, &c, the first figure on the right would express lines, the second inches, the third feet, the fourth fathoms, and the fifth dozens of fathoms, which might be expressed by a simple denomination, for example a perch, &c. In the last place, when it might be necessary to add, or subtract, or multiply, or divide similar quantities, we might operate as with whole numbers, and the result would in like manner express, according to the order of the figures, lines, inches, feet, &c.

It may easily be conceived how convenient this would be in practice. On this account Stevin, a Dutch mathematician, proposed to adapt the subdivisions of weights and measures to our present system of numeration, by making them decrease in decimal progression. According to this plan, the fathom would have contained 10 feet, the foot 10 inches, the inch 10 lines, &c. But he did not reflect on the inconvenience of depriving himself of the advantage of being able to divide his measures &c, by 3, 4 and 6, without a fraction, which is indeed of some importance.

It is evident that in the duodenary arithmetic, the nine first numbers might be expressed as usual, by the nine known characters 1, 2, 3, &c; but as the period ought to terminate only at twelve, it would be necessary to express ten and eleven by simple characters. In this case we might choose φ to denote ten, and ϑ to denote eleven, and then it is evident that,

10	would express twelve
11	. . . . thirteen
12	. . . . fourteen
13	. . . . fifteen
14	. . . . sixteen
15	. . . . seventeen
16	. . . . eighteen
17	. . . . nineteen
18	. . . . twenty
19	. . . . twenty-one
1φ	. . . . twenty-two
1ϑ	. . . . twenty-three
20	. . . . twenty-four
30	. . . . thirty-six
40	. . . . forty-eight

50	seventy-two
100	a hundred and forty-four
200	two hundred and eighty-eight
300	four hundred and thirty-two
1000	one thousand seven hundred and twenty-eight
2000	three thousand four hundred and fifty-six
10000	twenty thousand seven hundred and thirty-six
100000 &c.	two hundred and forty-eight thousand eight hundred and thirty-two

Thus the number denoted by the figures φ943 would be eighteen thousand six hundred and twenty-seven; for φ000 is eighteen thousand two hundred and eighty, 900 is one thousand two hundred and ninety-six, 40 is forty-eight, and 3 is three, numbers which if added will form the above sum.

It would be easy to form a set of rules for this new arithmetic, similar to those of common arithmetic; but as it does not seem likely that this mode of calculation will ever be brought into general use, we shall confine ourselves to what has been already said on the subject, and only add, that we have seen a book, printed in Germany, in which the common rules of arithmetic were explained in all the systems, the binary, ternary, quaternary, and so on, to the duodenary inclusively.

CHAPTER II.

Of some Short Methods of performing Arithmetical Operations.

§ I.

Method of Subtracting several Numbers from several other given Numbers, without making partial Additions.

56243	A
84564	
3252	
26848	
2942	B
3654	
2308	
162003	

TO give the reader an idea of this operation, one example will be sufficient. Let it be proposed to subtract all the sums below the line at B, from all those above it at A. Add, in the usual manner, all the lower figures of the first column on the right, which will make 14, and subtract their sum from the next highest number of tens, or 20. Add the remainder 6 to the corresponding column above at A, and the sum total will be 23. Write down 3 at the bottom, and because there were here two tens, as before, there is nothing to be reserved or carried. Add, in like manner, the figures of the second lower column, which will amount to 9, and this sum taken from 10 will leave 1; add 1 therefore to the second column of the upper numbers, the sum of which will be 20; write down 0 at the bottom, and because there were here two tens, while in the lower column there was only one, reserve the difference, and subtract it from the next column of the numbers marked B before you begin to add. In the contrary case, that is to say, when there are more tens in any one of the columns marked B than in the corresponding column above it, the difference must be added. In the last place, when it happens that this difference cannot be taken from the next column below, for want of more significant figures, as is the case here in the fifth column, we must add it to the upper one, and write down the whole sum below the line. By proceeding in this manner, we shall have, in the present instance, 162003 for the remainder of the subtraction required.

§ II.

Multiplication by the Fingers.

To multiply any two numbers, for example, 9 by 8; first take the difference between 9 and 10, which is 1, and having raised up the ten fingers of both hands, bend down 1 finger of one hand, for example the left. Then take the difference between 8 and 10 also, which is 2, and bend down 2 fingers of the right hand.

Count the fingers still raised up, which in this case are 7, and the sum will be the number of tens in the product. Multiply the number of the fingers bent down of one hand, by that of the fingers bent down of the other, and the result will be the number of units in the product. By this operation it will be found, in the present instance, that 9 multiplied by 8 makes 72.

It may hence be seen, that in general, we must take the difference between 10 and each of the given numbers; that the product of these differences, denoted by the fingers bent down of each hand, will give the units of the product, and that the number of the fingers which remain raised up, will give that of the tens of the same product.

It is evident however, that this operation is rather curious than useful; for no numbers but such as are less than ten can be multiplied in this manner, and every person almost can tell these first products from memory alone, otherwise they could not perform any complex multiplication at all.

§ III.

Some Short Methods of performing Multiplication and Division.

I. Every one, in the least acquainted with arithmetic, knows, that to multiply any number by 10, nothing is necessary but to add to it a cipher; that to multiply by 100, two ciphers must be added; and so on.

Hence it follows, that to multiply by 5, we have only to suppose a cipher added to the number, and then to divide it by 2. Thus, if it were required to multiply 127 by 5; suppose a cipher added to the former, which will give 1270, and then divide by 2: the quotient 635 will be the product required.

In like manner, to multiply any number by 25, we must suppose it multiplied by 100, or increased by two ciphers, and then divide by 4. Thus, 127 multiplied by 25, will give 3175. For 127 when increased by two ciphers makes 12700, which being divided by 4, produces 3175.

According to the same principle, to multiply by 125, it will be sufficient to add three ciphers to the multiplicand, or to suppose them added, and then to divide by 8. The reason of these operations may be so readily conceived, that it is not necessary to explain it.

II. The multiplication of any number by 11 may be reduced to simple addition. For it is evident that to multiply a number by 11, is nothing else than to add the number to its decuple, that is to say, to itself followed by a cipher.

Let the proposed number, for example be . 67583

To multiply this number by eleven say 3 and 0 743413

make 3; write down 3 in the units place; then add 8 and 3, which make 11; write down 1 in the place of tens, and carry 1; then 5 and 8 and 1 carried make 14; write down 4 in the third place, or that of hundreds, and carry 1. Continue in this manner, adding every figure to its next following one, till the operation is finished, and the product will be 743413, as above.

The same number may be multiplied in like manner, by 111, if we first write down the 3, then the sum of 8 and 3, then that of 5, 8, and 3, then that of 7, 5, and 8, and so on, adding always three contiguous figures together.

III. We shall only just further observe, that to multiply any number by 9, simple subtraction may be employed. Let us take, for example, the same number as before, 67583

608247

To multiply this number by 9, nothing is necessary but to suppose a cipher added to the end of it, and then to subtract each figure from that which precedes it, beginning at the right. Thus 3 from 0 or 10, leaves 7; 8 from 2 or 12, leaves 4; and if we continue in this manner, taking care to borrow 10 when the right-hand figure is too small to admit of the preceding one being subtracted from it, we shall find the product to be 608247.

The reason of these operations may be readily perceived. For it is evident, that in the first, we only add the number itself to its decuple; and in the latter, we subtract it from its decuple; but in order to form a clearer idea of the matter, it may perhaps be worth while to perform the operation at full length.

Concise operations of a similar kind may be employed in certain cases of division; as in dividing, for example, a given number by any power whatever of 5. Thus, if it were required to divide 128 by 5; we must double it, which will give 256; if we then cut off the last figure, which will be a decimal, the quotient will be 25·6 or $25\frac{6}{10}$. To divide the same number by 25, we must quadruple it, which will give 512; and if we then cut off the last two figures as decimals, we shall have for the quotient 5 and $\frac{12}{100}$. To divide by 125, we must multiply the dividend by 8, and cut off three figures. In like manner we may divide a given number by any other power of 5; but it must be confessed that such short methods of calculation are attended with no great advantage.

§ IV.

Short Method of performing Multiplication and Division by Napier's Rods or Bones.

When large numbers are to be multiplied, it is evident that the operation might be performed much readier, by having a table previously formed of each number of the multiplicand, when doubled, tripled, quadrupled, and so on. Such a table indeed might be procured by simple addition, since nothing would be necessary but to add any number to itself, and we should have the double; then to add it to the double, and we should have the triple, &c. But unless the same figure should frequently recur in the multiplicand, this method would be more tedious than that which we wished to avoid.

The celebrated Napier, the sole object of whose researches seems to have been to shorten the operations of arithmetic and trigonometry, and to whom we are indebted for the ingenious and ever-memorable invention of logarithms, devised a method of forming a table of this kind in a moment, by means of certain rods, which he has described in his work entitled *Rabdologia*, printed at Edinburgh in 1617. The construction of them is as follows:

Provide several slips of card or ivory or metal rods, about nine times as long as they are broad, and divide each of them into 9 equal squares. (*Plate* I. *fig.* 1.) Inscribe at the top, that is to say in the first square of each slip or rod, one of the numbers of the natural series 1, 2, 3, 4, &c, as far as 9 inclusively. Then divide each of the lower squares into two parts by a diagonal, drawn from the upper angle on the right hand to the lower one on the left, and inscribe in each of these triangular divisions, proceeding downwards, the double, triple, quadruple, &c, of the number inscribed at the top; taking care, when the multiple consists of only one figure, to place it in the lower triangle, and when it consists of two, to place the units in the lower triangle, and the tens in the upper one, as seen in the figure. It will be necessary to have one of these slips or rods the squares of which are not divided by a diagonal, but inscribed with the natural numbers from 1 to 9. This one is called the index rod. It will be proper also to have several of these slips or rods for each figure.

The rods being prepared as above, let us suppose that it is required to multiply the number 6785399. Arrange the seven rods inscribed at the top with the figures 6785399, close to each other, and apply to them on the left hand the index rod, or that inscribed with the single figures (*see Pl. I. fig.* 2.); by which means we shall have a table of all the multiples of each figure in the multiplicand; and scarcely any thing more will be necessary, but to transcribe them. Thus, for example, to multiply the above number by 6; looking for 6 on the index rod, and opposite to it in the first square, on the right hand, we find 54; writing down the 4 found in the lower triangle, and adding the 5 in the upper one to the 4 in the lower triangle of the next square on the left, which makes 9; write down the 9, and then add the 5 in the upper triangle in the same square to the 8 in the lower triangle of the next one; and proceed in this manner, taking care to carry as in common addition, and we shall find the result to be 40712394, or the product of 6785399 multiplied by 6.

Compound multiplication, or by several figures, may be performed in the same manner, and with equal facility. Let us suppose, for example, that the same number is to be multiplied by 839938. Write down the multiplicand, and the multiplier below it in the usual manner; and as the first figure of the multiplier is 8, look for it in the index rod, and by adding the different figures in the triangles of the horizontal column opposite to it, the sum will be found to be 54283192, or the

```
      6785399
       839938
     --------
     54283192
    20356197
   61068591
  61068591
 20356197
54283192
-------------
5699314465262
```

product of the above number by 8, which must be written down. Then find the sum of the figures in the horizontal column opposite to 3, and write the sum down as before, but carrying it one place farther to the left. Continue in this manner till you have gone through all the figures of the multiplier, and if the several partial products be then added as usual, you will have the total product, as above expressed.

A similar artifice may be employed to shorten division, especially when large sums are to be often divided by the same divisor. Thus, for example, if the number 1492992 is to be divided by 432, and if the same divisor must frequently occur, construct, in the manner above described, a table of the multiples of 432, which will scarcely require any farther trouble than that of transcribing the numbers, as may be seen here on the left.

1	432
2	864
3	1296
4	1728
5	2160
6	2592
7	3024
8	3456
9	3888

```
1492992 (3456
1296
----
 1969
 1728
 ----
  2419
  2160
  ----
   2592
   2592
   ----
   0000
```

When this is done, it may be readily perceived, that since 432 is not contained in the first three figures of the dividend, some multiple of it must be contained in the first four figures, viz. 1492. To find this multiple, you need only cast your eye on the table, to observe that the next less multiple of 432 is 1296, which stands opposite to 3; write down 3 therefore in the quotient, and 1296 under 1492, then subtract the former from the latter, and there will remain 196, to which if you bring down the next figure of the dividend, the result will be 1969. By casting your eye again on the table, you will find that 1728, which stands opposite to 4, is the greatest multiple of 432 contained in 1969; write down 4 therefore in the quotient, and subtract as before. By continuing the operation in this manner, it will be found that the following figures of the quotient are 5 and 6; and as the last multiple leaves no remainder, the division is perfect and complete.

Remark.

Mathematicians have not confined themselves to endeavouring to simplify the operations of arithmetic by such means: they have attempted something more, and have tried to reduce them to mere mechanical operations. The celebrated Pascal was the first who invented a machine for this purpose, a description of which may be seen in the fourth volume of the *Recueil des Machines preséntées à l'Academie.* Sir Samuel Morland, without knowing perhaps what Pascal had done in this respect, published, in 1673, an account of two arithmetical machines, which he invented, one of them for addition and subtraction, and the other for multiplication, but without explaining their internal construction. The same object engaged the attention of the celebrated Leibnitz, about the same time; and afterwards that of the marquis Poleni. A description of their machines may be seen in the *Theatrum Arithmeticum* of Leupold, printed in 1727, together with that of a machine invented by Leupold himself, and also in the *Miscel. Berol.* for 1709. We have likewise the *Abaque rabdologique* of Perrault, in the collection of his machines published in 1700. It serves for addition, subtraction, and multiplication. The *Recueil des Machines présentées à l'Academie Royale des Sciences* contains also an arithmetical machine, by Lespine, and three by Boistissandeau. Finally, Mr. Gersten, professor of mathematics at Giessen, transmitted, in the year 1735, to the Royal Society of London, a minute description of a machine of the same kind, invented by himself. We shall not enlarge farther on this subject, but proceed to give an account, which we hope will be acceptable to the curious reader, of an ingenious method of performing the operations of arithmetic, invented by Mr. Saunderson, a celebrated mathematician, who was blind from his infancy.

§ V.

Palpable Arithmetic, or a method of performing arithmetical operations, which may be practised by the blind, or in the dark.

WHAT is here announced may, on the first view, appear to be a paradox; but it is certain that this method of performing arithmetical operations was practised by the celebrated Dr. Saunderson, who, though he had lost his sight when a child of a year old, made so great progress in the mathematics, as to be able to fill a professor's chair in the university of Cambridge. The apparatus he employed to supply the deficiency of sight, was as follows.

Let the square ABCD (Pl. 3 fig. 2.) be divided into four other squares, by two lines parallel to the sides, and intersecting each other in the centre. These two lines form with the sides of the square four points of intersection, and these added to the four angles of the primitive square, give altogether, with the centre, nine points. If a hole be made in each of these points, into which a pin or peg can be fixed, it is evident that there will be nine distinct places for the nine simple and significant figures of our arithmetical system, and nothing further will be necessary but to establish some order in which these points or places, destined to receive a moveable peg, ought to be counted. To mark 1, it may be placed in the centre; to express 2, it may be placed immediately above the centre; to express 3, at the upper angle on the right; and so on in succession, round the sides of the square, as marked by the numbers opposite to each point. Pl. 3 fig. 1.

But there is still another character to be expressed, viz. the 0, which in our arithmetic is of very great importance. This character might be expressed in a manner exceedingly simple, by leaving the holes empty; but Saunderson preferred placing in the middle one a large-headed pin, unless when having unity to express, he was obliged to substitute in its stead a small-headed pin. By these means he obtained the advantage of being better able to direct his hands, and to distinguish with more ease, by the relative position of the small-headed pins, in regard to the large one in the centre, what the former expressed. This method therefore ought to be adopted; for Saunderson no doubt made choice of those means which were most significant to his fingers.

As the reader has here seen with what ease a simple number may be expressed in this manner, we shall now show that a compound number may be expressed with equal facility. If we suppose several squares to be constructed like the preceding, ranged in a line, and separated from each other by small intervals, that they may be better distinguished by the touch, any person acquainted with common arithmetic may perceive, that the first square on the right will serve to express units; the next towards the left, to express tens; the third to express hundreds, &c. Thus the five squares, with the pegs arranged as represented Pl. 3 fig. 4, will express the number 54023.

If you therefore provide a board, or table, divided into several horizontal bands, on each of which are placed seven or eight similar squares, according to circumstances; if these bands be separated by proper intervals, that they may be better distinguished; and if all the squares of the same order, in each of the bands, be so arranged, as to correspond to each other in a perpendicular direction; you may perform, by means of this machine, all the different operations of arithmetic. The reader will find, Plate 3 fig. 4, a representation of the method of adding four numbers, and expressing their sum by a machine of this kind.

Saunderson employed this ingenious machine, not only for arithmetical operations, but also for representing geometrical figures, by arranging his pins in a certain order, and extending threads from the one to the other. But what has been said is sufficient on this subject; those persons who are desirous of farther information respecting it, may consult Saunderson's Algebra, or the French translation of Wolf's Elements Abridged, where this palpable arithmetic is explained at full length.

PROBLEM.

To multiply 11£. 11*s. and* 11*d. by* 11£. 11*s. and* 11*d.*

THIS problem was once proposed by a sworn accountant, to a young man who had been recommended to him as perfectly well acquainted with arithmetic. And indeed, besides the difficulty which results from the multiplication of quantities of different kinds, and from their reduction, it is well calculated to try the ingenuity of an arithmetician. But it is not improbable that the proposer would have been embarrassed by the following simple question: what is the nature of the product of pounds shillings and pence multiplied by pounds shillings and pence? We know that the product of a yard by a yard represents a square yard, because geometricians have agreed to give that appellation to a square surface one yard in length and one in breadth; and 6 yards multiplied by 4 yards make 24 square yards; for a rectangular superficies 6 yards in length and 4 in breadth, contains 24 square yards, in the same manner as the product of 4 by 6 contains 24 units. But who can tell what the product of a penny by a penny is, or of a penny by a pound?

The question considered in this point of view, is therefore absurd, though ordinary arithmeticians sometimes are not sensible of it.

It may however be considered under different points of view, which will render it susceptible of a solution. The first is to observe that a pound contains 20 shillings, or 240 pence; so that the problem may be reduced to the following, in abstract numbers: to multiply 11 plus $\frac{11}{20}$ plus $\frac{11}{240}$ by 11 plus $\frac{11}{20}$ plus $\frac{11}{240}$; and in this case, the product will be 134 plus $\frac{9}{20}$ plus $\frac{3}{240}$ plus $\frac{49}{57600}$.

The second way of considering this question, is to observe that every product is the fourth term of a proportion, the first term of which is unity, while the two quantities to be multiplied are the second and third terms. Nothing therefore is necessary but to fix that kind of unity which ought to be the first term of the proportion.

We may say, for example, if a pound employed in any way has produced 11£. 11*s.* and 11*d.* how much will 11£. 11*s.* and 11*d.* produce? The product here will be the same as above, viz. 134*l.* 9*s.* 3*d.* and $\frac{49}{240}$ of a penny.

But this unit might be a shilling, for there is nothing to prevent us from expressing the question in this manner: If a shilling produce 11£. 11*s.* and 11*d.*, how much ought 11£. 11*s.* and 11*d.* to produce? The product then would be 2689*l.* 5*s.* 4*d.* and $\frac{1}{12}$ of a penny.

In the last place, this unit might be a penny; and in that case the product will be 32271*l.* 4*s.* 1*d.*

CHAPTER III.

Of certain Properties of Numbers.

WE do not here mean to examine those properties of numbers which engaged so much the attention of the ancients, and in which they pretended to find so many mysterious virtues. Every one, whose mind is not tinctured with the spirit of credulity, must laugh to think of the good canon of Cezene, Peter Bungus, collecting in a large quarto volume entitled *De Mysteriis Numerorum*, all the ridiculous ideas which Nichomachus, Ptolemy, Porphyry, and several more of the ancients, childishly propagated respecting numbers. How could it enter the minds of reasonable beings, to ascribe physical energy to things entirely metaphysical? For numbers are mere conceptions of the mind, and consequently can have no influence in nature.

None therefore but old women and people of weak minds can believe in the virtues of numbers. Some imagine, that if thirteen persons sit down at the same table, one of them will die in the course of the year; but there is a much greater probability that one will die if the number be twenty-four.

I.

THE number 9 possesses this property, that the figures which compose its multiples, if added together, are always a multiple of 9; so that by adding them, and rejecting 9 as often as the sum exceeds that number, the remainder will always be 0. This may be easily proved by trying different multiples of 9, such as 18, 27, 36, &c.

This observation may be of utility to enable us to discover whether a given number be divisible by 9; for in all cases, when the figures which express any number, on being added together, form 9, or one of its multiples, we may be assured that the number is divisible by 9; and consequently by 3 also.

But this property does not exclusively belong to the number 9; for the number 3 has a similar property. If the figures which express any multiple of 3 be added, we shall find that their sum is always a multiple of 3, and when any proposed number is not such a multiple, whatever the sum of the figures by which it is expressed exceed a multiple of 3, will be the quantity to be deducted from the number, in order that it may be divisible by 3 without a remainder.

We must not omit to take notice here, of a very ingenious observation of the author of the History of the Academy of Sciences, for the year 1726, which is, that if a system of numeration different from that now in use had been adopted, that for example of duodecimal progression, the number eleven, or, in general, that preceding the first period, would have possessed the same property as the number nine does in our present system of numeration. By way of example let us take a multiple of eleven, as nine hundred and fifty-seven, and let us express it according to that system by the characters 7φ5: it will here be seen that 7 and φ make seventeen, and 5 added makes twenty-two, which is a multiple of eleven.

We shall not here attempt to demonstrate why this property belongs to the last number but one, of the period adopted for numeration, as it would lead us into analytical researches of too complex a nature.

In addition to the foregoing observations of the French author, may be added the following remarks on the same subject, lately made by an ingenious English gentleman. He first expresses all the products of 9 by the other figures, in the following manner, and then enumerates the curious properties.

```
9
1
—
9..9
 2
 —
 18..1+8=9
         3
        ——
        27..2+7=9
                4
               ——
               36..3+6=9
                       5
                      ——
..6+3=9               45..4+5=9
    8                         6
   ——                        ——
   72..7+2=9                 54..5+4=9
           9                         7
           —                         —
           81..8+1=9.                63..
```

The component figures of the product, made by the multiplication of every digit into the number 9, when added together, make NINE.

The order of those component figures is reversed, after the said number has been multiplied by 5.

The component figures of the amount of the multipliers, (viz. 45), when added together make NINE.

The amount of the several products, or multiples of 9 (viz. 405), when divided by 9, gives for a quotient, 45; that is 4+5=NINE.

The amount of the first product (viz. 9), when added to the other products, whose respective component figures make 9, is 81; which is the *square of* NINE.

The said number 81, when added to the above-mentioned amount of the several products, or multiples of 9 (viz. 405), makes 486; which, if divided by 9, gives for a quotient 54; that is 5+4=NINE.

It is also observable that the number of changes that may be rung on *nine* bells, is 362880; which figures, added together, make 27; that is, 2+7=NINE.

And the quotient of 362880, divided by 9, is 40320; that is 4 + 0 + 3 + 2 + 0 = NINE.

II.

EVERY square number necessarily ends with one of these five figures, 1, 4, 5, 6, 9; or with an even number of ciphers preceded by one of these figures. This may be easily proved, and is of great utility in enabling us to discover when a number is not a square; for though a number may end as above mentioned, it is not always however a perfect square; but, at any rate, when it does not end in that manner, we are certain that it is not a square, which may prevent useless labour. In regard to cubic numbers, they may end with any figure whatever; but if they terminate with ciphers, they must be in number either three, or six, or nine, &c.

If a square number terminate with a 4, the last figure but one will be an even number: as in 64, and 144, and 97344, &c.

If a square number terminate with 5, it will terminate with 25: or 625, or 1225, or 2025, &c.

If a square number terminate with an odd digit, the last figure but one will be even; as in the squares 81, 529, 3721, 6889, &c. But if it terminate with any even digit, except 4, the last figure but one will be odd; as in these squares, 36, 576, 1936, 13456, &c.

No square number can terminate with two equal digits, except two ciphers, or two fours; as 100, 144, 40000, 44944, &c.

A square number cannot terminate in three equal digits, except they be three fours; as 213444. Nor can it terminate in more than three digits, unless they be ciphers; as 90000.

III.

EVERY square number is divisible by 3, or becomes so when diminished by unity. This may be easily tried on any square number at pleasure.—Thus 4 less 1, 16 less 1, 25 less 1, 121 less 1, &c. are all divisible by 3; and the case is the same with other square numbers.

Every square number is divisible also by 4, or becomes so when diminished by unity. This may be proved with the same ease as the former.

Every square number is divisible likewise by 5, or becomes so when increased, or else diminished by unity. Thus, for example, $36-1$, $49+1$, $64+1$, $81-1$, &c. are all divisible by 5.

Every odd square number is a multiple of 8 increased by unity. We have examples of this property in the numbers 9, 25, 49, 81, &c; from which if 1 be deducted, the remainders will be divisible by 8.

If a square number be either multiplied or divided by a square, the product or quotient will be a square.

If a number be not a complete square, its square root cannot be expressed, either by an integer, or by a rational fraction, either proper or improper.

IV.

Every number is either a square, or divisible into two or three or four squares. Thus 30 is equal to $25+4+1$; $31=25+4+1+1$; $33=16+16+1$; $63=49+9+4+1$, or $=36+25+1+1$.

We shall here add, by anticipation, though we have not yet informed the reader what triangular, or pentagonal, &c, numbers are, that

Every number is either triangular, or composed of two or of three triangular numbers. And that

Every number is either pentagonal, or composed of two or three or four or five pentagonals, and so of the rest.

We shall add also, that every even square, after the first square 1, may be resolved at least into four equal squares; and that every odd square may be resolved into three, if not into two. Thus 81=36+36+9; 121=81+36+4; 169=144+25; 625=400+144+81.

V.

Every power of 5, or of 6, necessarily ends with 5 or with 6.

VI.

If we take any two numbers whatever; then either one of them, or their sum, or their difference, is necessarily divisible by 3. Let the numbers assumed be 20 and 17; though neither of these numbers, nor their sum 37, is divisible by 3, yet their difference is, for it is 3.

It might easily be demonstrated, that this must necessarily be the case, whatever be the numbers assumed.

VII.

If two numbers are of such a nature, that their squares when added together form a square, the product of these two numbers is divisible by 6.

Of this kind, for example, are the numbers 3 and 4, the squares of which, 9 and 16, when added, make the square number 25: their product 12 is divisible by 6.

From this property a method may be deduced, for finding two numbers, the squares of which, when added together, shall form a square number. For this purpose, multiply any two numbers together; the double of their product will be one of the numbers sought, and the difference of their squares will be the other.

Thus if we multiply together 2 and 3, the squares of which are 4 and 9, their product will be 6; if we then take 12, the double of this product, and 5 the difference of their squares, we shall have two numbers, the sum of whose squares is equal to another square number; for these squares are 144 and 25, which when added make 169, the square of 13.

VIII.

When two numbers are such, that the difference of their squares is a square number; the sum and difference of these numbers are themselves square numbers, or the double of square numbers.

Thus, for example, the numbers 13 and 12, when squared, give 169 and 144, the difference of which, 25, is also a square number; then 25, the sum of these numbers, is a square number, and also their difference 1.

In like manner, 6 and 10, when squared produce 36 and 100, the difference of which 64 is also a square number; then it will be found, that their sum 16 is a square number, as well as their difference 4.

The numbers 8 and 10 give for the difference of their squares 36; and it may be readily seen, that 18, the sum of these numbers, is the double of 9, which is a square number, and that their difference 2 is the double of 1, which is also a square number.

IX.

If two numbers, the difference of which is 2, be multiplied together, their product increased by unity will be the square of the intermediate number.

Thus, the product of 12 and 14 is 168, which being increased by 1, gives 169, the square of 13, the mean number between 12 and 14.

Nothing is easier than to demonstrate, that this must always be the case; and it will be found in general, that the product of two numbers increased by the square of half their difference, will give the square of the mean number.

X.

A *prime* number, is that which has no other divisor but unity. Numbers of this kind, the number 2 excepted, can never be even, nor can any of them terminate in 5, except 5 itself; hence it follows, that except those contained in the first period of ten, they must necessarily terminate in 1 or 3 or 7 or 9.

One curious property of prime numbers is, that every prime number, 2 and 3 excepted, if increased or diminished by unity, is divisible by 6. This may be readily seen in any numbers taken at pleasure, as 5, 7, 11, 13, 17, 19, 23, 29, 31, &c; but I do not know, that any one has ever yet demonstrated this property *a priori*. But the inverse of this is not true, that is, every number which when increased or diminished by unity is divisible by 6, is not, on that account, necessarily a prime number.

The product arising from two different prime numbers, cannot be a square number.

As it is often of utility to be able to know, without having recourse to calculation, whether a number be prime or not, we have here subjoined a table of all the prime numbers from 1 to 10,000.

TABLE

Of the prime numbers from 1 *to* 10,009.

2	193	449	733	1033	1327	1657	1999	2339
3	197	457	739	1039	1361	1663	2003	2341
5	199	461	743	1049	1367	1667	2011	2347
7	211	463	751	1051	1373	1669	2017	2351
11	223	467	757	1061	1381	1693	2027	2357
13	227	479	761	1063	1399	1697	2029	2371
17	229	487	769	1069	1409	1699	2039	2377
19	233	491	773	1087	1423	1709	2053	2381
23	239	499	787	1091	1427	1721	2063	2383
29	241	503	797	1093	1429	1723	2069	2389
31	251	509	811	1097	1433	1733	2081	2393
37	257	521	821	1103	1439	1741	2083	2399
41	263	533	823	1109	1447	1747	2087	2411
43	269	541	827	1117	1451	1753	2089	2417
47	271	547	829	1123	1453	1759	2099	2423
53	277	557	839	1129	1459	1777	2111	2437
59	281	563	853	1151	1471	1783	2113	2441
61	283	569	857	1153	1481	1787	2129	2447
67	293	571	859	1163	1483	1789	2131	2459
71	307	577	863	1171	1487	1801	2137	2467
73	311	587	877	1181	1489	1811	2141	2473
79	313	593	881	1187	1493	1823	2143	2477
83	317	599	883	1193	1499	1831	2153	2503
89	331	601	887	1201	1511	1847	2161	2521
97	337	607	907	1213	1523	1861	2179	2531
101	347	613	911	1217	1531	1867	2203	2539
103	349	617	919	1223	1543	1871	2207	2543
107	353	619	929	1229	1549	1873	2213	2549
109	359	631	937	1231	1553	1877	2221	2551
113	367	641	941	1237	1559	1879	2237	2557
127	373	643	947	1249	1567	1889	2239	2579
131	379	647	953	1259	1571	1901	2243	2591
137	383	653	967	1277	1579	1907	2251	2593
139	389	659	971	1279	1583	1913	2267	2609
149	397	661	977	1283	1597	1931	2269	2617
151	401	673	983	1289	1601	1933	2273	2621
157	409	677	991	1291	1607	1949	2281	2633
163	419	683	997	1297	1609	1951	2287	2647
167	421	691	1009	1301	1613	1973	2293	2657
173	431	701	1013	1303	1619	1979	2297	2659
179	433	709	1019	1307	1621	1987	2309	2663
181	439	719	1021	1319	1627	1993	2311	2671
191	443	727	1031	1321	1637	1997	2333	2677

2688	3041	3457	3821	4217	4621	5009	5437	5827
2687	3049	3461	3823	4219	4637	5011	5441	5839
2689	3061	3463	3833	4229	4639	5021	5443	5843
2693	3067	3467	3847	4231	4643	5023	5449	5849
2699	3079	3469	3851	4241	4649	5039	5471	5851
2707	3083	3491	3853	4243	4651	5051	5477	5857
2711	3089	3499	3863	4253	4657	5059	5479	5861
2713	3109	3511	3877	4259	4663	5077	5483	5867
2719	3119	3517	3881	4261	4673	5081	5501	5869
2729	3121	3527	3889	4271	4679	5087	5503	5879
2731	3137	3529	3907	4273	4691	5099	5507	5881
2741	3163	3533	3911	4283	4703	5101	5519	5897
2749	3167	3539	3917	4289	4721	5107	5521	5903
2753	3169	3541	3919	4297	4723	5113	5527	5923
2767	3181	3547	3923	4327	4729	5119	5531	5927
2777	3187	3557	3929	4337	4733	5147	5557	5939
2789	3191	3559	3931	4339	4751	5153	5563	5953
2791	3203	3571	3943	4349	4759	5167	5569	5981
2797	3209	3581	3947	4357	4773	5171	5573	5987
2801	3217	3583	3967	4363	4787	5179	5581	6007
2803	3221	3593	3989	4373	4789	5189	5591	6011
2819	3229	3607	4001	4391	4793	5197	5623	6029
2833	3251	3613	4003	4397	4799	5209	5639	6037
2837	3253	3617	4007	4409	4801	5227	5641	6043
2843	3257	3623	4013	4421	4813	5231	5647	6047
2851	3259	3631	4019	4423	4817	5233	5651	6053
2857	3271	3637	4021	4441	4831	5237	5653	6067
2861	3299	3643	4027	4447	4861	5261	5657	6073
2879	3301	3659	4049	4451	4871	5273	5659	6079
2887	3307	3671	4051	4457	4877	5279	5669	6089
2897	3313	3673	4057	4463	4889	5281	5683	6091
2903	3319	3677	4073	4481	4903	5297	5689	6101
2909	3323	3691	4079	4483	4909	5303	5693	6113
2917	3329	3697	4091	4493	4919	5309	5701	6121
2927	3331	3701	4093	4507	4931	5323	5711	6131
2939	3343	3709	4099	4513	4933	5333	5717	6133
2953	3347	3719	4111	4517	4937	5347	5737	6143
2957	3359	3727	4127	4519	4943	5351	5741	6151
2963	3361	3733	4129	4523	4951	5381	5743	6163
2969	3371	3739	4133	4547	4957	5387	5749	6173
2971	3373	3761	4139	4549	4967	5393	5779	6197
2999	3389	3767	4153	4561	4969	5399	5783	6199
3001	3391	3769	4157	4567	4973	5407	5791	6203
3011	3407	3779	4159	4583	4987	5413	5801	6211
3019	3413	3793	4177	4591	4993	5417	5807	6217
3023	3433	3797	4201	4597	4999	5419	5813	6221
3037	3449	3803	4211	4603	5003	5431	5821	6229

6247	6661	7043	7523	7917	8363	8779	9203	9623
6257	6673	7057	7529	7927	8369	8783	9209	9629
6263	6679	7069	7537	7933	8377	8803	9221	9631
6269	6689	7079	7541	7937	8387	8807	9227	9643
6271	6691	7103	7547	7949	8389	8819	9239	9649
6277	6701	7109	7549	7951	8419	8821	9241	9661
6287	6703	7121	7559	7963	8423	8831	9257	9677
6299	6709	7127	7561	7993	8429	8837	9277	9679
6301	6719	7129	7573	8009	8431	8839	9281	9689
6311	6733	7151	7577	8011	8443	8849	9283	9697
6317	6737	7159	7583	8017	8447	8861	9293	9719
6323	6761	7177	7589	8039	8461	8863	9311	9721
6329	6763	7187	7591	8053	8467	8867	9319	9733
6337	6779	7193	7603	8059	8501	8887	9323	9739
6343	6781	7207	7607	8069	8513	8893	9337	9743
6353	6791	7211	7621	8081	8521	8923	9341	9749
6359	6793	7213	7639	8087	8527	8929	9343	9767
6361	6803	7219	7643	8089	8537	8933	9349	9769
6367	6823	7229	7649	8093	8539	8941	9371	9781
6373	6827	7237	7669	8101	8543	8951	9377	9787
6379	6829	7243	7673	8111	8563	8963	9391	9791
6389	6833	7247	7681	8117	8573	8969	9399	9803
6397	6841	7253	7687	8123	8581	8971	9403	9811
6421	6857	7283	7691	8147	8597	8999	9413	9817
6427	6863	7297	7699	8161	8599	9001	9419	9829
6449	6869	7307	7703	8167	8609	9007	9421	9833
6451	6871	7309	7717	8171	8623	9011	9431	9839
6469	6883	7321	7723	8179	8627	9013	9433	9851
6473	6899	7331	7727	8191	8629	9027	9437	9857
6481	6907	7333	7741	8209	8641	9041	9439	9859
6491	6911	7349	7753	8219	8647	9043	9461	9871
6521	6917	7351	7757	8221	8663	9049	9463	9883
6529	6947	7369	7759	8231	8669	9059	9467	9887
6547	6949	7393	7789	8233	8677	9067	9473	9901
6551	6959	7411	7793	8237	8681	9091	9479	9907
6553	6961	7417	7817	8243	8689	9103	9491	9923
6563	6967	7433	7823	8263	8693	9109	9497	9929
6569	6971	7451	7829	8269	8699	9127	9511	9931
6571	6977	7457	7841	8273	8707	9133	9521	9941
6577	6983	7459	7853	8287	8713	9137	9533	9949
6581	6991	7477	7867	8291	8719	9151	9539	9967
6599	6997	7481	7873	8293	8731	9157	9547	9973
6607	7001	7487	7877	8297	8737	9161	9551	
6619	7013	7489	7879	8311	8741	9173	9587	
6637	7019	7499	7883	8317	8747	9181	9601	
6653	7027	7507	7901	8329	8753	9187	9613	
6659	7039	7517	7907	8353	8761	9199	9619	

XI.

Another kind of numbers which possess a singular and curious property, are those called *perfect numbers*. This name is given to every number, the aliquot parts of which, when added together, form exactly that number itself. Of this we have an example in the number 6; for its aliquot parts are 1, 2, 3, which together make 6. The number 28 possesses the same property; for its aliquot parts are 1, 2, 4, 7, 14, the sum of which is 28.

To find all the perfect numbers of the numerical progression, take the double progression 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, &c; and examine those terms of it, which, when diminished by unity, are prime numbers. Those to which this property belongs, will be found to be 4, 8, 32, 128, 8192; for these terms when diminished by unity, are 3, 7, 31, 127, 8191. Multiply therefore each of these numbers by that number in the geometrical progression which preceded the one from which it is deduced, for example 3 by 2, 7 by 4, 31 by 16, 127 by 64, 8191 by 4096, &c; and the result will be 6, 28, 496, 8128, 33550336, which are perfect numbers.

These numbers however are far from being so numerous as some authors have believed*. The following is a series of numbers either perfect, or, for want of proper attention, supposed to be so, taken from a memoir of Mr. Krafft, published in the 7th volume of the Transactions of the Academy of Petersburg. Those to which this property really belongs are marked with an asterisk.

* The rule given by Ozanam is false, and produces a multitude of numbers, such as 130816, 2096128, &c, which are not perfect numbers. When Ozanam wrote his rule, he did not recollect, that one of the multipliers must be a prime number. But 511 and 2047 are not prime numbers.

* 6
* 28
* 496
* 8128
130816
2096128
* 33550336
536854528
* 8589869056
* 137438691328
2199022206976
35184367894528
562949936644096
9007199187632128
144115187807420416
* 2305843008139952128
36893488143124135936

Thus we find that between 1 and 10 there is only one perfect number; that there is one between 10 and 100, one between 100 and 1000, and one between 1000 and 10000; but those would be mistaken who should believe that there is one also between ten thousand and a hundred thousand, one between a hundred thousand and a million, &c; for there is only one between ten thousand and eight hundred millions. The rarity of perfect numbers, says a certain author, is a symbol of that of perfection.

All the perfect numbers terminate with 6 or 28, but not alternately or conversely.

XII.

THERE are some numbers called *amicable numbers*, on account of a certain property which gives them a kind of affinity or reciprocity, and which consists in their being mutually equal to the sum of each others aliquot parts. Of this kind are the numbers 220 and 284; for the first

220 is equal to the aliquot parts of 284, viz. 1, 2, 4, 71, 142; and, reciprocally, 284 is equal to the aliquot parts, 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110, of the other number 220.

Amicable numbers may be found by the following method. Write down, as in the subjoined example, the terms of a double geometrical progression, or having the ratio 2, and beginning with 2; then triple each of these terms, and place these triple numbers each under that from which it has been formed; these numbers diminished by unity, 5, 11, 23, &c, if placed each over its corresponding number in the geometrical progression, will form a third series above the latter. In the last place, to obtain the numbers of the lowest series 71, 287, &c, multiply each of the terms of the series, 6, 12, 24, &c, by the one preceding it, and subtract unity from the product.

5	11	23	47	95	191	383
2	4	8	16	32	64	128
6	12	24	48	96	192	384
	71	287	1151	4607	18431	73727

Take any number of the lowest series, for example 71, of which its corresponding number in the first series, viz. 11, and the one preceding the latter, viz. 5, as well as 71, are prime numbers: multiply 5 by 11, and the product 55 by 4, the corresponding term of the geometrical series, and the last product 220, will be one of the numbers required. The second will be found by multiplying the number 71 by the same number 4, which will give 284.

In like manner, with 1151, 47 and 23, which are prime numbers, we may find two other amicable numbers, 17296 and 18416; but 4607 will not produce any amicable numbers, because, of the two other corresponding numbers, 47 and 95, the latter is not a prime number. The case is the same with the number 18431, because 95 is among its corresponding numbers; but the following number 73727, with 383 and 191, will give two more amicable numbers, 9363584 and 9437056.

By these examples it may be seen, that if perfect numbers are rare, amicable numbers are much more so, the reason of which may be easily conceived.

XIII.

If we write down a series of the squares of the natural numbers, viz. 1, 4, 9, 16, 25, 36, 49, &c; and take the difference between each term and that which follows it, and then the differences of these differences; the latter will each be equal to 2, as may be seen in the following example.

	1	4	9	16	25	36	49
1st. Diff.	3	5	7	9	11	13	
2d. Diff.	2	2	2	2	2		

It hence appears, that the square numbers are formed by the continual addition of the odd numbers 1, 3, 5, &c, which exceed each other by 2.

In the series of the cubes of the natural numbers, viz. 1, 8, 27, &c, the third, instead of the second differences, are equal, and are always 6, as may be seen in the following example.

Cubes	1	8	27	64	125	216
1st. Diff.	7	19	37	61	91	
2d. Diff.	12	18	24	30		
3d. Diff.	6	6	6			

In regard to the series of the fourth powers, or biquadrates, of the natural numbers, the fourth differences only are equal, and are always 24. In the fifth powers, the fifth differences only are equal, and are invariably 120.

These differences, 2, 6, 24, 120, &c, may be found by multiplying the series of the numbers 1, 2, 3, 4, 5, 6, &c. For the second power, multiply the first two; for the third power, the first three, and so on.

XIV.

THE progression of the cubes 1, 8, 27, 64, 125, &c, of the natural numbers, 1, 2, 3, 4, 5, 6, &c, possesses this remarkable property, that if any number of its terms whatever, from the beginning, be added together, their sum will always be a square. Thus, 1 and 8 make 9; if we add to this sum 27, we shall have 36, which is still a square number; and if we add 64, we shall have 100, and so on.

The root of each square thus constituted, is always equal to the sum of the roots of all the component cubes. As, the sum of 1, 8, 27, 64, 125, or of 1^3, 2^3, 3^3, 4^3, 5^3, is $225 = 15^2$ the square of $1 + 2 + 3 + 4 + 5$.

XV.

THE number 120 has the property, of being equal to half the sum of its aliquot parts, or divisors, viz. 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, which together make 240. The number 672 is also equal to half the sum of its aliquot parts, 1344. Several other numbers of the like kind may be found, and some even which would form only a third, or fourth, of the sum of their aliquot parts, or which would be the double, triple, or quadruple of that sum; but what has been here said, will be sufficient to exercise those who are fond of such researches.

CHAPTER IV.

Of Figurate Numbers.

IF there be taken any arithmetical progression, as for instance, the most simple of all, or that of the natural numbers, 1, 2, 3, 4, 5, 6, 7, &c; and if we take the first term, the sum of the first two, that of the first three, and so on; the result will be a new series of numbers, 1, 3, 6, 10, 15, 21, 28, &c, called *triangular numbers*, because they can always be ranged in such a manner as to form an equilateral triangle, as may be seen *Plate* 1 *fig.* 3.

The square numbers, as 1, 4, 9, 16, 25, 36, &c, arise from a like addition of the first terms of the arithmetical progression, 1, 3, 5, 7, 9, 11, &c, the common difference of which is 2. These numbers, as is well known, may be arranged so as to form square figures. See Pl. 1 fig. 4.

A similar addition of the terms of the arithmetical progression 1, 4, 7, 10, 13, &c, the common difference of which is 3, will produce the numbers 1, 5, 12, 22, &c; which are called pentagonal numbers, because they represent the number of points which may be arranged on the sides and in the interior part of a regular pentagon; as may be seen *Plate* 1 *fig.* 5; where there are three pentagons, having one common angle, representing the number of points which increase arithmetically; the first having two points on each side, the second three, and the third four; and which progression it is evident, might be continued ever so far.

It is in this sense, and in this manner, that we must conceive the figurate numbers to be arranged.

It is almost needless to say, that the progression 1, 5, 9, 13, 17, &c, the common difference of which is 4, produces, by a similar addition, the hexagonal numbers, which are 1, 6, 15, 28, 45, &c; and that in like manner may be found the heptagonals, the octagonals, &c.

There is another kind of polygonal numbers, which result from the number of points that can be ranged in the middle, and on the sides, of one or more similar polygons, having a common centre. These are different from the preceding; for the series of the triangulars of this kind is 1, 4, 10, 19, 31, &c, which are formed by the successive addition of the numbers, 1, 3, 6, 9, 12.

The central square numbers are 1, 5, 13, 25, 41, 61, &c; formed, in like manner, by the successive addition of the numbers 1, 4, 8, 12, 16, 20, &c.

The central pentagonal numbers are 1, 6, 16, 31, 51, 76, &c; formed by the addition of the numbers 1, 5, 10, 15, 20, &c.

But we shall not enlarge further on this kind of polygonal numbers, because they are not those to which mathematicians usually give that name. Let us return therefore to the ordinary polygonal numbers.

The radix of a polygonal number, is the number of the terms of the progression necessary to be added in order to obtain that number. Thus, the radix of the triangular number 21, is 6, because that number results from the successive addition of the six numbers 1, 2, 3, 4, 5, 6. In like manner, 4 is the radix of the square number 16, considered as a figurate number, because that number is produced by adding the *four* terms 1, 3, 5, 7, of the progression of the odd numbers.

Having given this explanation of the nature of polygonal numbers, we shall now present the reader with a few problems respecting them.

PROBLEM I.

To find whether any proposed Number is Triangular, or Square, or Pentangular, &c.

The method of finding whether a number be square, is well known, and serves as a foundation for discovering the other figurate numbers. This being supposed; then to determine whether any given number is a polygonal number, the following general rule may be employed.

Multiply by 8 the number of the angles of the polygon less 2; multiply this first product by the proposed number, and to the new product add the square of a number equal to that of the angles of the polygon less 4: if the sum be a perfect square, the given number is a polygon of the kind proposed.

It may easily be seen, that as the number of the angles in the triangle are 3, in the square 4, in the pentagon 5, &c, we shall have, as the multiplier of the proposed number, in the case of the triangular number, 8; in that of the quadrangular number, 16; in that of the pentagonal, 24; and in that of the hexagonal, 32.

In like manner, as the number of the angles less 4, gives for the triangle — 1; for the square 0; for the pentagon 1; for the hexagon 2; &c; the numbers to be added to the product, as before mentioned, will be for the triangle 1 (because the square of — 1 is 1); for the square 0; for the pentagon, 1; for the hexagon, 4; for the heptagon, 9; &c. From these principles we may deduce the following rules, which we shall illustrate by examples.

Suppose it were required to know whether 21 be a triangular number.

Multiply 21 by 8, to the product add 1, and the sum will be 169, which is a perfect square: consequently 21 is a triangular number.

If we are desirous of knowing whether 35 be a pentagonal number, we must multiply 35 by 24, and the product will be 840; to this product if 1 be added, we shall have 841, which is a square number: we may therefore rest assured that 35 is a pentagonal number.

PROBLEM II.

A Triangular, or any Figurate Number whatever, being given; to find its Radix, or the Number of the Terms of the Arithmetical Progression of which it is the Sum.

First perform the operation described in the preceding problem; and having found the square root, the possibility of which will indicate whether the number be figurate or not; add to this root a number equal to that of the angles of the proposed polygon less 4, and divide the sum by the double of the same number of angles less 2: the quotient will be the radix of the polygon.

The number to be added is, for the triangle —1, that is to say 1 to be deducted; for the square it is 0; for the pentagon 1; for the hexagon 2; &c.

As to the divisor, it may be easily seen that for the triangle it is 2 (because the double of 3 less 2 is 2), for the square 4, for the pentagon 6, for the hexagon 8, &c.

Let it be required therefore to find the radix of the triangular number 36.

Having performed the operation explained in the preceding problem, and found the product 289, the square root of which is 17, subtract unity from this number, and divide the remainder by 2; the quotient 8 will be the radix or side of the triangular number 36.

Let the radix of the pentagonal number 35 be required.

Having found, as before, the radix 29, add to it 1, which will give 30, and divide by 6; the quotient 5 will be the radix of this pentagonal number, that is to say, of the number formed by the addition of the 5 terms of the series 1, 4, 7, 10, 13.

PROBLEM III.

The Radix of a Polygonal Number being given; to find that Number.

The rule for this purpose is exceedingly simple. From the square of the given radix, subtract the product of the same radix by a number equal to that of the angles less four; the half of the remainder will be the polygonal number required.

For example, what is the triangular number the radix of which is 12?

The square of 12 is 144; the number equal to that of the angles less 4 is —1, which being multiplied by 12 gives—12: but according to the rule—12 ought to be subtracted, which is the same thing as adding 12; in that case you will have 156, which being divided by 2 gives 78.

What is the heptagonal number the radix of which is 20?

To find the number required, take the square of 20, which is 400; then multiply 20 by 3, which is the number of the angles less 4, and subtract 60, the product, from 400; if you then divide the remainder 340 by 2, the quotient 170 will be the number sought, or the heptagon the radix of which is 20.

It may not be improper here to remark, that the same number may be a polygon or figurate number in different ways. Every number greater than 3 is a polygon, of a number of sides or angles equal to that of its units.

Thus 36 is a polygon of 36 sides, the radix of which is 2; for the first two terms of the progression are 1, 35. The same number 36 is a square; and lastly, it is triangular, having 8 for its radix.

In the like manner, 21 is a polygon of 21 sides; it is also triangular; and lastly it is octagonal.

PROBLEM IV.

To find the Sum of as many Triangular, or of as many Square, or of as many Pentagonal Numbers, as we choose.

As by the successive addition of the terms of different arithmetical progressions, we obtain new progressions of numbers, called triangular numbers, square numbers, pentagonals, &c; we can add also these last progressions, which will give rise to new figurate numbers, of a higher order, called *pyramidal numbers.* Those which arise from the progression of triangular numbers, are called pyramidals of the first order; those produced by the addition of the square numbers, pyramidals of the second order; and those by the progression of the pentagonal numbers, pyramidals of the third order. The same operation may be performed with the pyramidals; which gives rise to the pyramido-pyramidals. But as these numbers are of little utility, and can answer no other purpose than that of exercising the genius of such as are fond of analytical investigation, we shall not enlarge farther on the subject. We shall therefore confine ourselves to giving a general rule for adding as many figurate numbers as the reader may choose.

Multiply the cube of the number of terms to be added, by the number of the angles of the polygon less 2; to the sum add three times the square of the said number of terms, and subtract from it the product of the same number multiplied by that of the angles less 5: if you then divide the remainder by 6, you will have the sum of the terms of the progression.

For example, suppose it were required to find the sum of the eight first triangular numbers.

The cube of 8 is 512; which being multiplied by the number of the angles of the polygon less 2, or by 1, gives still 512; add to this number the triple of the square of 8, or 192, which will make 704; then, as the number of the angles less 5, is −2, multiply 8 by −2, and you will have −16; if you then add 16 to 704, you will have 720; which being divided by 6, gives 120, for the sum of the eight first triangular numbers.

The same result may be obtained, with more ease, by multiplying the number of the terms 8, by 9, and the product by 10, which gives also 720; which divided by 6, the quotient is 120, as before.

In the case of a series of squares, the number of which we shall here suppose to be 10, we have only to multiply the number of terms, viz. 10, by the same number plus unity, or by 11, and then by the double of the same number plus unity, that is to say by 21: the product of these three numbers, 2310, if divided by 6, gives 385, for the sum of the first ten square numbers 1, 4, 9, 16, &c.

CHAPTER V.

Of Rightangled Triangles in Numbers.

A RIGHTANGLED triangle in numbers, consists of three numbers of such a nature, that the sum of the squares of two of them is equal to the square of the third. Of this kind, for example, are the three numbers 3, 4, 5, which express the simplest rightangled triangle of all; for, if 9 the square of 3, be added to 16 the square of 4, the sum will be 25, which is the square of 5. The numbers 3, 4, 5, express therefore the three sides of a rightangled triangle.

It may here be observed, that these numbers must necessarily be unequal; for if two of them were equal, they would be the two sides of a rightangled isosceles triangle: but it can be demonstrated that, in such cases, it is impossible to express the hypothenuse by a rational number, either whole or fractional, because a triangle of this kind is the half of a square, the two equal sides of the triangle being the sides, and the hypothenuse the diagonal; and it is well known that the diagonal of a square is incommensurable to the side.

It is necessary also that the three numbers which form the triangle should be rational numbers, either whole or fractional; otherwise it would require no art to find as many numbers of this kind as we might choose; for we would have nothing more to do, than to take any two numbers whatever, as 2 and 6, the sum of the squares of which is 40, and the hypothenuse would be $\sqrt{40}$; but $\sqrt{40}$ has no precise signification, and only shows that the square root of 40 must be extracted, which it is impossible in finite terms to do.

The area of a rightangled triangle, whose sides are rational, cannot be equal to a rational square.

If a, b, c represent the three sides of a triangle, and C the angle contained by the two sides a and b: then,

if $a^2+b^2=c^2$, the angle $C = 90°$,
if $a^2+ab+b^2=c^2$, - - $C = 120°$,
if $a^2-ab+b^2=c^2$, - - $C = 60°$.

Having made these previous observations, we shall now propose a few of the most curious and easy problems respecting rightangled triangles in numbers.

PROBLEM I.

To find as many Rightangled Triangles in Numbers, as we please.

Take any two numbers at pleasure, for example 1 and 2, which we shall call generating numbers; multiply them together; then having doubled the product, we obtain one of the sides of the triangle, which in this case will be 4. If we then square each of the generating numbers, which in the present example will give 4 and 1, their difference 3 will be the second side of the triangle, and their sum 5 will be the hypothenuse. The sides of the triangle, therefore, having 1 and 2 for their generating numbers, are 3, 4, 5.

If 2 and 3 had been assumed as generating numbers, we should have found the sides to be 5, 12, and 13; and the numbers 1 and 3 would have given 6, 8, and 10.

Another Method. Take a progression of whole and fractional numbers, as $1\frac{1}{3}$, $2\frac{2}{5}$, $3\frac{3}{7}$, $4\frac{4}{9}$, &c, the properties of which are: 1st. The whole numbers are those of the common series, and have unity for their common difference. 2d. The numerators of the fractions, annexed to the whole numbers, are also the natural numbers. 3d. The denominators of these fractions are the odd numbers, 3, 5, 7, &c.

Take any term of this progression, for example $3\frac{3}{7}$, and reduce it to an improper fraction, by multiplying the whole number 3 by 7, and adding to 21, the product, the numerator 3, which will give $\frac{24}{7}$. The numbers 7 and 24 will be the sides of a rightangled triangle, the hypothenuse of which may be found by adding together the squares of these two numbers, viz. 49 and 576, and extracting the square root of the sum. The sum in this case being 625, the square root of which is 25, this number will be the hypothenuse required. The sides therefore of the triangle produced by the above term of the generating progression, are 7, 24, 25.

In like manner, the first term $1\frac{1}{3}$ will give the rightangled triangle 3, 4, 5.

The second term $2\frac{2}{5}$ will give 5, 12, 13.

The fourth 4^4 will give 9, 40, 41. All these triangles have the ratio of their sides different; and they all possess this property, that the greatest side and the hypothenuse differ only by unity.

The progression $1\frac{7}{8}$, $2\frac{11}{12}$, $3\frac{15}{16}$, $4\frac{19}{20}$, &c, is of the same kind as the preceding. The first term of it gives the rightangled triangle 8, 15, 17; the second term gives the triangle 12, 35, 37; the third the triangle 16, 63, 65; &c. All these triangles, it is evident, are different in regard to the proportion of their sides; and they all have this peculiar property, that the difference between the greater side and the hypothenuse, is always the number 2.

PROBLEM II.

To find any Number of Rightangled Triangles in Numbers, the Sides of which shall differ only by Unity.

To resolve this problem, we must find out such numbers that the double of their squares plus or minus unity shall also be square numbers. Of this kind are the numbers 1, 2, 5, 12, 29, 70, &c; for twice the square of 1 is 2, which diminished by unity leaves 1, a square number. In like manner, twice the square of 2 is 8, to which if we add 1, the sum 9 will be a square number. And so on.

Having found these numbers, take any two of them which immediately follow each other, as 1 and 2, or 2 and 5, or 12 and 29, for generating numbers. The rightangled triangles arising from them will be of such a nature, that their sides will differ from each other only by unity. The following is a table of these triangles, with their generating numbers.

Gener. Numb.		Sides.		Hypoth.
1	2	3	4	5
2	5	20	21	29
5	12	119	120	169
12	29	696	697	985
29	70	4059	4060	5741
70	169	23660	23661	33461

But if the problem were, to find a series of triangles of such a nature, that the hypothenuse of each should exceed one of the sides only by unity, the solution would be much easier. Nothing in this case would be necessary but to assume, as the generating numbers of the required triangle, any two numbers having unity for their difference. The following is a table, similar to the preceding, of the first six rightangled triangles produced by the first numbers of the natural series.

Gener. Numb.		Sides.		Hypoth.
1	2	3	4	5
2	3	5	12	13
3	4	7	24	25
4	5	9	40	41
5	6	11	60	61
6	7	13	84	85

If we assume, as generating numbers, the respective sides of the preceding series of triangles, we shall have a new series of rightangled triangles, the hypothenuses of which will always be square numbers; as may be seen in the following table.

Gener. Numb.		Sides.		Hypoth.	Roots.
3	4	7	24	25	5
5	12	119	120	169	13
7	24	336	527	625	25
9	40	720	1519	1681	41
11	60	1320	3479	3721	61
13	84	2184	6887	7225	85

It may here be observed, that the roots of the hypothenuses are always equal to the greater of the generating numbers increased by unity.

But if the second side and the hypothenuse of each triangle in the above table, which differ only by unity, were assumed as the generating numbers, we should have a series of rightangled triangles, the least sides of which would always be squares. A few of these are as follow:

Gener. Numb.		Sides.		Hypoth.
4	5	9	40	41
12	13	25	312	313
24	25	49	1200	1201
40	41	81	3280	3281

In the last place, if it were required to find a series of rightangled triangles, one of the sides of which shall be always a cube, we have nothing to do but to take, as generating numbers, two following terms in the progression of triangular numbers, as 1, 3, 6, 10, 15, 21, &c. By way of example we shall here give the first four of these triangles:

Gener.	Numb.	Sides.		Hypoth
1	3	6	8	10
3	6	36	27	45
6	10	120	64	136
10	15	300	125	326

PROBLEM III.

To find Three different Rightangled Triangles, the Areas of which shall be all Equal.

The following are three rightangled triangles which possess this property. The sides of the first are, 40, 42, 48; those of the second 24, 70, 74; and those of the third, 15, 112, 113.

The method in which these triangles are found, is as follows:

Add the product of any two numbers to the sum of their squares, and that will give the first number; the difference of their squares will give a second; and double the sum of their product and of the square of the least number, will give the third.

If you then form a rightangled triangle from the two first of the numbers thus found, as generating numbers; a second from the two extremes; and a third from the first and the sum of the other two; these three rightangled triangles will be equal to each other.

No more than three rightangled triangles, equal to each other, can be found in whole numbers; but we may find as many as we choose in fractions or mixed numbers, by means of the following formula:

With the hypothenuse of one of the above triangles, and the quadruple of its area, form another rightangled triangle, and divide it by double the product which arises from multiplying the hypothenuse of the triangle you made choice of by the difference of the squares of the two other sides: the triangle thence produced will be the one required.

PROBLEM IV.

To find a Rightangled Triangle, the Sides of which shall be in Arithmetical Progression.

Take two generating numbers which have to each other the ratio of 1 to 2; the sides of the rectangled triangle thence produced will be in arithmetical progression.

The simplest of these triangles, is that which has for its sides 3, 4 and 5, arising from the numbers 1 and 2 assumed as generating numbers. But it is to be observed, that all the other triangles, which possess the same property, are similar to this one, and are only multiples of it. That there can be no other kind, might easily be demonstrated in a great many different ways.

REMARK.

If it were required to find a rightangled triangle, the three sides of which should be in geometrical proportion, we must observe that none such can be found in whole numbers; for the two generating numbers ought to be in the ratio of 1 to $\sqrt{(\sqrt{5}-2)}$, which is an irrational number.

PROBLEM V.

To find a Rightangled Triangle, the Area of which, expressed in Numbers, shall be equal to the Perimeter, or in a given ratio to it.

Of any square number, and the same square increased by 2, form a rightangled triangle, and divide each of its sides by that square number: the quotients will give the sides of a new rightangled triangle, the area of which, expressed numerically, will be equal to the perimeter.

Thus, if we take, as generating numbers, 1 and 3, we shall have the triangle 6, 8, 10, the sides of which if divided by unity give the same 6, 8, 10, forming a triangle having the property required; for the area and the perimeter are each equal to 24. In like manner, if we take 2 and 6 as generating numbers, we obtain for the required triangle 5, 12, 13, which on trial will be found to possess the same property.

These triangles are the only two of the kind which can be found in whole numbers; but we may find abundance of them in fractional numbers, by means of the squares 9, 16, &c; such as the following: $\frac{40}{9}$, $\frac{198}{9}$, $\frac{202}{9}$; or $\frac{68}{16}$, $\frac{576}{16}$, $\frac{580}{16}$, or in their least terms, $\frac{17}{4}$, $\frac{144}{4}$, $\frac{145}{4}$.

If it were required that the area of the proposed triangle should be only in a given ratio to the perimeter, for example that of $\frac{3}{2}$; take as generating numbers a square and the same square increased by 3, and from them, as before directed, form a rightangled triangle: this triangle will possess the required property. Of this kind, in whole numbers, are the two triangles 8, 15, 17, and 7, 24, 25: and numberless others may be found in fractional numbers.

CHAPTER VI.

Some Curious Problems respecting Square and Cube Numbers.

PROBLEM I.

Any Square Number being given, to divide it into Two other Squares.

INNUMERABLE solutions may be found to this problem, in the following manner. Let 16, for example, whose root is 4, be the square to be divided into two other squares, which, as may be easily seen, can only be fractions.

Take any two numbers, as 3 and 2; multiply them together; and by their product multiply the double of 4, the root of the proposed square; the last product, which in this case is 48, will be the numerator of a fraction, the denominator of which will be 13, the sum of the squares of the above numbers 3 and 2: the fraction $\frac{48}{13}$ therefore will give the side of the first square required, which square consequently will be $\frac{2304}{169}$.

To obtain the second, multiply the given square 16, by the above denominator 169, and from the product 2704, subtract the numerator 2304: if we then take 20, the root of 400 the remainder, (which will be always a square), for a numerator, and 13 for a denominator, we shall have the fraction $\frac{20}{13}$ for the side of the second square.

The two sides of the required squares therefore, are $\frac{48}{13}$ and $\frac{20}{13}$, the squares of which, $\frac{2304}{169}$ and $\frac{400}{169}$, will be found equal to the square number 16.

If we had taken for the primitive numbers 2 and 1, we should have had the roots $\frac{16}{5}$ and $\frac{12}{5}$, the squares of which are $\frac{256}{25}$ and $\frac{144}{25}$; the sum of which is $\frac{400}{25}$ or 16.

The numbers 4 and 3 would have given the roots $\frac{96}{25}$ and $\frac{28}{25}$, the squares of which $\frac{9216}{625}$ and $\frac{784}{625}$ still make up $\frac{10000}{625}$ or 16.

It may here be seen, that by varying the first two supposed numbers at pleasure, the solutions also may be varied without end.

REMARK.

Should it here be asked whether a given cube can, in like manner, be divided into two other cubes? We shall reply, on the authority of an eminent analyst, M. de Fermat, that it is not possible. It is equally impossible to divide any power above the square into two parts, which shall be powers of the same kind; for example, a biquadrate into two biquadrates.

PROBLEM II.

To divide a Number, which is the Sum of Two Squares, into Two other Squares.

Let the proposed number be 13, which is composed of the two squares 9 and 4: it is required to divide it into two other squares.

Take any two numbers, for example, 4 and 3; and multiply the former, 4, by 6, the double of 3 the root of one of the above squares; and the second 3 multiply by the double of 2 the root of the other square; which will give as products 24 and 12. Subtract the latter of these numbers from the former, and their difference 12 will be the numerator of a fraction, the denominator of which will be 25, the sum of the squares of the numbers first assumed. Multiply this fraction $\frac{12}{25}$, by each of the assumed numbers, viz. 4 and 3, and you will have $\frac{48}{25}$ and $\frac{36}{25}$. If you then take the greater of these numbers from the root of the greater square contained in 13, viz. 3, the remainder will be $\frac{27}{25}$; and if you add the other to the side of the smaller square contained in 13, viz. 2, you will have $\frac{86}{25}$. These two fractions then, $\frac{27}{25}$ and $\frac{86}{25}$, will be the sides of the two squares sought, viz. $\frac{729}{625}$ and $\frac{7396}{625}$, which together are equal to 13, as may be easily proved.

By supposing other numbers, other squares may be obtained; but these are sufficient to show the method of finding them.

REMARK.

For a number to be divisible, in a variety of ways, into two squares, it must be either a square, or composed of two squares. Of this kind, taking them in order, are the numbers 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 25, 26, 29, 32, 34, 36, 37, &c. We do not know, nor do we think it possible to find, any method of dividing into two squares, any number which is not a square, or the sum of two squares; and we are of opinion that it may be established as a rule, that every whole number, which is not a square or composed of two squares, in whole numbers, cannot be divided, in any manner, into two squares. A demonstration of this would be curious.

But every number is divisible, in a great variety of ways, into four squares; for there is no number which is not either a square, or the sum of two, or of three, or of four squares. Bachet de Meziriac advanced this proposition *, the truth of which he ascertained as far as possible by trying all the numbers from 1 to 325. It is added, by M. de Fermat †, that he was able to demonstrate the following general and curious properties of numbers, viz.

That every number is either triangular, or composed of two or three triangular numbers.

* Diophanti Alexandrini Arithmeticorum lib. vi. cum Comm. C. G. Bacheti. Tolosæ. 1670. fol. p. 179. † Ibid. p. 180.

That every number is either square, or composed of two, or three, or four square numbers.

And that every number is either pentagonal, or composed of two, or three, or four or five pentagonal numbers. And so of the rest.

A demonstration of these properties of numbers, if they be real, would be truly curious.

PROBLEM III.

To find Four Cubes, two of which taken together, shall be equal to the Sum of the other two.

This problem may be solved by the following simple method. Take any two numbers of such a nature, that double the cube of the less shall exceed that of the greater; then from double the greater cube subtract the less; and multiply the remainder, as well as the sum of the cubes, by the lesser of the assumed numbers: the two products will be the sides of the two first cubes required.

In like manner, take the cube of the greater of the assumed numbers from double the cube of the less; and multiply the remainder, as well as the sum of these two cubes, by the greater of the assumed numbers: the two new products will be the sides of the other two cubes.

For example, if we assume the numbers 4 and 5, which possess the above property, we obtain, by following the rule, for the sides of the two first cubes, 744, 756; and for those of the other two, 945 and 15, which being divided by 3, give for the two first 248, 252; and for the two latter, 315, 5.

If the assumed numbers be 5 and 6, we shall have 1535 and 1705 for the sides of the two first cubes; and 2046 and 204 for those of the other two.

REMARK.

A number composed of two cubes being given, it is possible to find two other cubes, the sum of which shall be equal to the former two. Vieta was of a contrary opinion; but M. de Fermat, in his Observations on the Arithmetical Questions of Diophantus, with a commentary by Bachet de Meziriac, has pointed out a method by which such cubes can be found. The calculation indeed extends to numbers which are exceedingly complex, and sufficient to frighten the boldest arithmetician; as may be seen by the following example, where it is required to divide the sum of the two cubes 8 and 1 into two other cubes. By following the method of M. de Fermat, Father de Billy found that the sides of the two new cubes were the following fractional numbers:

$$\frac{12436177733990097836481}{6096238356613729744 9},$$

and

$$\frac{4872671717143523365 60}{6096238356613729744 9}.$$

We must take these numbers on Father de Billy's word; for we do not know that any one will ever venture to examine whether he has been deceived.

But it is possible to resolve, without much trouble, another question of a similar kind, which is: to find three cubes which, taken together, shall be equal to a fourth. By following the method pointed out in the above mentioned work, it will be found that the least whole numbers, which resolve the question, are 3, 4 and 5; for their cubes added together make 216, which is the cube of 6.

We have confined ourselves to a few questions of this kind, but they might be varied almost without end. They are attended with a peculiar kind of difficulty which renders them interesting, and on that account they have been an object of attention to various analysts; such as Diophantus of Alexandria, among the ancients, who wrote thirteen books on arithmetical questions, of which the first six only remain, with another on polygonal numbers.

Vieta too exercised his ingenuity on questions of this kind; as did also Bachet de Meziriac, who wrote a commentary on the above work of the Greek Arithmetician. But this species of analysis was carried farther than ever it had been before by the celebrated M. de Fermat. Father de Billy, about the same time, gave proofs of the acuteness of his talents in this way, by his work entitled *Diophantus Redivivus*, in which he far excelled the ancient analyst. M. Ozanam likewise showed great ability in this species of analysis, by the resolution of several problems, which had been considered as insoluble. He wrote a work on this subject, but it was never published; and the manuscript, after his death, came into the hands of the late M. Daguesseau, as we are informed by the historian of the Academy of Sciences *.

CHAPTER VII.

Of Arithmetical and Geometrical Progressions, and of certain Problems which depend on them.

§ I.

Explanation of the most remarkable properties of an Arithmetical progression.

IF there be a series of numbers, either increasing or decreasing, in such a manner, that the difference between the first and the second, shall be equal to that between the second and third, and between the third and fourth, and so on successively; these numbers will be an arithmetical progression.

* The Hindus must also be considered as having been great adepts in such kind of problems; as we learn from some of their algebraical works that have lately been found among them; of which an account is given in the second volume of my Tracts, lately printed. C. H.

The series of numbers 1, 2, 3, 4, 5, 6, &c; or 1, 5, 9, 13, &c; or 20, 18, 16, 14, 12, &c; or 15, 12, 9, 6, 3, are therefore arithmetical progressions; for in the first, the difference between each term and the following one, which exceeds it, is always 1; in the second it is 4: in like manner this difference is always 2 in the third series, which goes on decreasing, and in the fourth it is 3.

It may be readily seen, that an increasing arithmetical progression may be continued ad infinitum; but this cannot be the case, in one sense, with a decreasing series; for we must always arrive at some term, from which if the common difference be taken, the remainder will be 0, or else a negative quantity. Thus, the progression 19, 15, 11, 7, 3, cannot be carried farther, at least in positive numbers; for it is impossible to take 4 from 3, or if it be taken we shall have, according to analytical expression, −1*; and by continuing the subtraction we should have −5, −9, &c.

The chief properties of arithmetical progressions may be easily deduced from the definition which we have here given. For a little attention will show,

1st. That each term is nothing else than the first, plus or minus the common difference multiplied by the number of intervals between that term and the first. Thus, in the progression, 2, 5, 8, 11, 14, 17, &c, the difference of which is 3, there are five intervals between the sixth term and the first; and for this reason the sixth term is equal to the first plus 15, the product of the common difference 3 by 5. But as the number of intervals is always less by unity than the number of terms, it thence follows, that we may

* As the quantities called negative are real quantities, taken in a sense contrary to that of the quantities called positive, it is evident that, according to mathematical and analytical strictness, an arithmetical progression may be continued ad infinitum, decreasing as well as increasing; but we here speak agreeably to the vulgar mode of expression only.

find any term, the place of which in the series is known, if we multiply the common difference by the number expressing that place less unity. According to this rule, the hundredth term of an increasing progression will be equal to the first plus 99 times the common difference. If it be decreasing, it will be equal to the first term minus that product.

In every arithmetical progression therefore, the common difference being given, to find any term the place of which is known; multiply the common difference by the number which indicates that place less unity, and add the product to the first term, if the progression be increasing, but subtract it if it be decreasing: the sum or remainder will be the term required.

2d. In every arithmetical progression, the sum of the first and last terms, is equal to that of the second and the last but one; and to that of the third and the last but two; &c; in short, to the sum of the middle terms if the number of the terms be even, or to the double of the middle term if the number of the terms be odd.

This may easily be demonstrated from what has been said: for let us call the first term A, and let us suppose that there are twenty terms in the progression; if it be increasing, the twentieth term will be equal to A plus nineteen times the common difference; and their sum will be double the first term plus 19 times that difference. But the second term is equal to the first plus the common difference, and the 19th term, or last but one, according to our supposition, is equal to the first plus eighteen times that difference. The sum therefore of the second and last but one, is twice the first term plus 19 times the common difference, the same as before. And so of the third and last but two.

3d. By this last property we are enabled to show, in what manner the sum of all the terms of an arithmetical progression may be readily found: for, as the first and last terms make the same sum as the second and last but one, and as the third and the last but two, &c; in short, as the two middle terms, if the number of terms be even; it thence follows, that the whole progression contains as many times the sum of the first and the last terms, as there are pairs of such terms. But the number of pairs is equal to half the number of the terms; consequently the sum of the whole progression, is equal to the product of the sum of the first and last terms multiplied by half the number of terms, or, what amounts to the same, to half the product of the sum of the first and the last terms by the number of the terms of the progression.

If the number of the terms be odd, as 9 for example; it may be readily seen that the middle term will be equal to half the sum of the two next to it, and consequently of the sum of the first and the last. But the sum of all the terms, the middle term excepted, is equal to the product of the sum of the first and last terms by the number of terms less unity, for example 8 in the case here proposed, where there are 9 terms; consequently, by adding the middle term, which will complete the sum of the progression, and which is equal to half the sum of the first and the last terms, we shall have, for the sum total of the progression, as many times the half sum above mentioned, as there are terms in the progression; which is the same thing as the product of half the sum of the first and last terms by the number of the terms, or the product of the whole sum by half the number of terms.

When these rules are well understood, it will be easy to resolve the following questions.

PROBLEM I.

If a hundred stones are placed in a straight line, at the distance of a yard from each other; how many yards must *the person walk, who undertakes to pick them up one by one, and to put them into a basket a yard distance from the first stone?*

It is evident that, to pick up the first stone, and put it into the basket, the person must walk 2 yards, one in going and another in returning; that for the second he must walk 4 yards; and so on, increasing by two as far as the hundredth, which will oblige him to walk two hundred yards, one hundred in going, and one hundred in returning. It may easily be perceived also, that these numbers form an arithmetical progression, in which the number of terms is 100, the first term 2, and the last 200. The sum total therefore, will be the product of 202 by 50, or 10100 yards, which amount to more than five miles and a half.

PROBLEM II.

A gentleman employed a bricklayer to sink a well, and agreed to give him at the rate of three shillings for the first yard in depth, five for the second, seven for the third, and so on increasing till the twentieth, where he expected to find water: how much was due to the bricklayer when he had completed the work?

This question may be easily answered, by the rules already given; for the difference of the terms is 2, and the number of terms 20; consequently, to find the twentieth term, we must multiply 2 by 19, and add 38, the product, to the first term 3, which will give 41 for the twentieth term.

If we then add the first and last terms, that is 3 and 41, which will make 44, and multiply this sum by 10, or half the number of terms, the product 440 will be the sum of all the terms of the progression, or the number of shillings due to the bricklayer when he had completed the work. He would therefore have to receive 22*l.*

PROBLEM III.

A gentleman employed a bricklayer to sink a well to the depth of 20 yards, and agreed to give him 20l. for the whole; but the bricklayer falling sick, when he had finished the eighth yard, was unable to go on with the work: how much was then due to him?

Those who might imagine that two fifths of the whole sum were due to the workman, because 8 yards are two fifths of the depth agreed on, would certainly be mistaken; for it may be easily seen that, in cases of this kind, the labour increases in proportion to the depth. We shall here suppose, for it would be difficult to determine it with any accuracy, that the labour increases arithmetically as the depth; consequently the price ought to increase in the same manner.

To determine this problem, therefore, 20*l.* or 400 shillings must be divided into 20 terms in arithmetical progression, and the sum of the first eight of these will be what was due to the bricklayer for his labour.

But 400 shillings may be divided into twenty terms, in arithmetical proportion, a great many different ways, according to the value of the first term, which is here undetermined: if we suppose it, for example, to be 1 shilling, the progression will be 1, 3, 5, 7, &c, the last term of which will be 39; and consequently the sum of the first eight terms will be 64 shillings. On the other hand, if we suppose the first term to be $10\frac{1}{2}$, the series of terms will be $10\frac{1}{2}$, $11\frac{1}{2}$, $12\frac{1}{2}$, $13\frac{1}{2}$, $14\frac{1}{2}$, which will give 112 shillings for the sum of the first eight terms.

But, to resolve the problem in a proper manner, so as to give to the bricklayer his just due for the commencement of the work, we must determine what is the fair value of a yard of work, similar to the first, and then assume that value as the first term of the progression. We shall here suppose that this value is 5 shillings; and in that case the required progression will be 5, $6\frac{11}{19}$, $8\frac{3}{19}$, $9\frac{14}{19}$, $11\frac{6}{19}$, $12\frac{17}{19}$, &c, the common difference of which is $\frac{30}{19}$, and the last term 35. Now to find the eighth term, which is necessary before we can find the sum of the first eight terms, multiply the common difference $\frac{30}{19}$ by 7, which will give $11\frac{1}{19}$, and add this product to 5 the first term, which will give the eighth term $16\frac{1}{19}$; if we then add $16\frac{1}{19}$ to the first term, and multiply the sum, $21\frac{1}{19}$, by 4, the product, $84\frac{4}{19}$, will be the sum of the first eight terms, or what was due to the bricklayer, for the part of the work he had completed. The bricklayer therefore had to receive $84\frac{4}{19}$ shillings, or 4*l.* 4*s.* $2\frac{10}{19}$*d.*

PROBLEM IV.

A merchant being considerably in debt, one of his creditors, to whom he owed 1860*l. offered to give him an acquittance if he would agree to pay the whole sum in* 12 *monthly installments; that is to say,* 100*l. the first month, and to increase the payment by a certain sum each succeeding month, to the twelfth inclusive, when the whole debt would be discharged: By what sum was the payment of each month increased?*

In this problem the payments to be made each month ought to increase in arithmetical progression. We have given the sum of the terms, which is equal to the sum total of the debt, and also the number of these terms, which is 12; but their common difference is unknown, because it is that by which the payments ought to increase each month.

To find this difference, we must take the first payment multiplied by the number of terms, that is to say 1200 pounds, from the sum total, and the remainder will be 660;

we must then multiply the number of terms less unity, or 11, by half the number of terms, or 6, and we shall have 66; by which if the remainder 660 be divided, the quotient 10 will be the difference required. The first payment therefore being 100, the second payment must have been 110, the third 120, and the last 210.

§ II.

Of Geometrical Progressions, with an explanation of their Principal Properties.

If there be a series of terms, each of which is the product of the preceding by a common multiplier; or, what amounts to the same thing, each of which is in the same ratio to the preceding; such a series forms what is called a geometrical progression. Thus 1, 2, 4, 8, 16, &c, form a geometrical progression; for the second is the double of the first, the third the double of the second, and so on in succession. The terms 1, 3, 9, 27, 81, &c, form also a geometrical progression, each term being the triple of that which precedes it.

I. The principal property of a geometrical progression is, that if we take any three following terms, as 3, 9, 27, the product 81, of the extremes, will be equal to the square of the middle term 9; in like manner, if we take four following terms, as 3, 9, 27, 81, the product of the extremes, 243, will be equal to the product of the two means or middle terms, 9 and 27.

In the last place, if we take any successive number of terms, as 2, 4, 8, 16, 32, 64, the product of the extremes, 2 and 64, will be equal to the product of any two which are equally distant from them, viz. 4 and 32, or 8 and 16. If the number of the terms were odd, it is evident that there would be only one term equally distant from the two extremes; and in that case, the square of this term would be equal to the product of the extremes, or to that of any two equally distant from them, or from the mean term.

II. Between geometrical and arithmetical progression there is a certain analogy, which deserves here to be mentioned, and which is, that the same results are obtained in the former by employing multiplication and division, as are obtained in the latter by addition and subtraction. When in the latter we take the half or the third, we employ in the former extraction of the square, cube, &c, roots.

Thus, to find an arithmetical mean between any two numbers, for example 3 and 12, we add the two given extremes, and $7\frac{1}{2}$, the half of their sum 15, will be the number required; but to find a geometrical mean between the two numbers, we must multiply the two extremes, and extract the square root of their product. Thus, if the given numbers were 3 and 12, by extracting the square root of their product 36, we shall have 6 for the number required.

If we take any geometrical progression whatever, as, 1, 2, 4, 8, 16, 32, 64, &c, and write it down as in the subjoined example, with the terms of an arithmetical progression above it, in regular order,

0	1	2	3	4	5	6	7	8	9	10
1	2	4	8	16	32	64	128	256	512	1024

the following properties will be remarked in this combination:

1st. If any two terms whatever of the geometrical progression, for example 4 and 64, be multiplied together, their product will be 256; if we then take the two terms of the arithmetical progression corresponding to 4 and 64, which are 2 and 6, and add them together, their sum 8 will be found over the above sum 256.

2d. If we take four terms of the lower series in geometrical proportion, for example, 2, 16, 64, 512, the numbers of the upper series corresponding to them will be 1, 4, 6, 9, which are in arithmetical proportion; for the difference between 4 and 1 is the same as that between 9 and 6.

3d. In the lower series, if we take any square number, for example 64, and in the upper series the term corresponding to it, viz. 6, the half of the latter will be found to correspond to the square root of 64, the former, viz. 8.

By taking, in the lower series, a cube, for example 512, and in the upper series the corresponding number 9, it will be found that the third of the latter, which is 3, will correspond to the cube root of the former 512, which is 8.

Thus, it is evident, that what is multiplication in geometrical progression, is addition in arithmetical; that what is division in the former, is subtraction in the latter; and, in the last place, that what is extraction of the square, cube, &c, roots, in geometrical progression, is simple division by 2, 3, &c, in arithmetical.

This remarkable analogy is the foundation of the common theory of logarithms; and on that account, seemed worthy of some illustration.

III. It is evident that all the powers of the same number, taken in regular order, form a geometrical progression; as may be seen in the following example, which is a series of the powers of the number 2,

2 4 8 16 32 64 128 &c.

The case is the same with the powers of the number 3, which form the series,

3 9 27 81 243 729 &c.

The first of these series has this peculiar property, that if we take the first, second, fourth, eighth, sixteenth, and thirty-second terms, and to them, add unity, the result will be prime numbers.

IV. The common ratio of a geometrical progression, is the number that results from the division of any term by that which precedes it. Thus, in the geometrical progression 2, 8, 32, 128, 512, the ratio is 4; for if we divide 128 by 32, 32 by 8, or 8 by 2, the quotient will be always 4. The ratio therefore acts an important part in geometrical progression; the same that the common difference does in arithmetical, that is to say, it is always constant.

To find any term then, for example the 8th, of a geometrical progression, the ratio and first term of which are known, multiply the ratio by itself 7 times, or as many times as there are units in the place of the required term less one; or, what is the same thing, raise the ratio to the 7th power; then multiply the first term by the product, and the new product will be the 8th term required. For example, let the first term of the progression be 3, and the ratio 2; to find the 8th term, raise 2 to the 7th power, which will be 128, and multiply 128 by the first term 3, the product 384 will give the 8th term of the progression required.

We shall here observe, that had the 8th term of an arithmetical progression been required, the first term and the common difference being given, we should have multiplied that difference by 7, and added the product to the first term; which is a proof of the analogy already mentioned in the second paragraph.

V. The sum of the terms of any given geometrical progression may be found in the following manner.

Multiply the first term by itself, and the last by the second, and take the difference of the two products. Then divide this difference by that of the first two terms, and the quotient will be the sum of all the terms.

Let us take, for example, the progression 3, 6, 12, 24, &c, the eighth term of which is 384, and let it be required to find the sum of these eight terms: the product of the first term by itself is 9, and that of the last by the second is 2304; the difference of these products is 2295; if this difference then be divided by 3, the difference of the first and second terms, we shall have for quotient the number 765, which will be the sum of these eight terms.

VI. A geometrical progression may decrease *in infinitum*, without ever reaching 0; for it is evident that any part of the quantity greater than 0 can never become 0. A decreasing geometrical progression therefore may be extended without end; for by dividing the last term by the ratio of the progression, we shall have the following term.

We shall here subjoin two of these decreasing progressions, by way of examples:

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$, &c.

$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \frac{1}{729}$, &c.

VII. The sum of an increasing geometrical progression is evidently infinite; but that of a decreasing geometrical progression, whatever be the number of terms assumed, is always finite. Thus the sum of all the terms, *in infinitum*, of the progression $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, &c, is only 2; that of the progression $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$, &c, *in infinitum*, is only $1\frac{1}{2}$; &c. This necessarily follows from the method already given, for finding the sum of any number of terms whatever of a geometrical progression; for if we suppose it prolonged *in infinitum*, and decreasing, the last term will be infinitely small, or 0; the product of the second term by the last will therefore be 0; and consequently, to find the sum, nothing will be necessary but to divide the square of the first term by the difference of the first and the second. In this manner it will be found that the sum of $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, &c, continued *in infinitum*, is 2; and that of $1, \frac{1}{3}, \frac{1}{9}$, will be $\frac{3}{2}$ or $1\frac{1}{2}$; for the square of 1 is 1, the difference of 1 and $\frac{1}{2}$ is $\frac{1}{2}$, and unity divided by $\frac{1}{2}$ gives 2; in like manner, 1 being divided by $\frac{2}{3}$, which is the difference of 1 and $\frac{1}{3}$, gives $\frac{3}{2}$.

REMARK.

When we say that a progression continued *in infinitum* may be equal to a finite quantity, we do not, like Fontenelle, pretend to assert that infinity can have a real existence. What is here meant, and what ought to be understood, by all such expressions, is, that whatever be the number of terms of a progression assumed, their sum never can equal the determined finite quantity, though it may approach to it in such a manner, that their difference will become smaller than any assignable quantity.

PROBLEM I.

If Achilles can walk ten times as fast as a tortoise, which is a furlong before him, can crawl; will the former overtake the latter, and how far must he walk before he does so?

This problem has been thought worthy of notice merely because Zeno, the founder of the sect of the Stoics, pretended to prove by a sophism, that Achilles could never overtake the tortoise; for while Achilles, said he, is walking a furlong, the tortoise will have advanced the tenth of a furlong; and while the former is walking that tenth, the tortoise will have advanced the hundredth part of a furlong, and so on *in infinitum;* consequently an infinite number of instants must elapse before the hero can come up with the reptile, and therefore he will never come up with it.

Any person however, possessed of common sense, may readily perceive that Achilles will soon come up with the tortoise, since he will get before it. In what then consists the sophism? It may be explained as follows:

Achilles indeed would never overtake the tortoise if the intervals of time during which he is supposed to be walking the first furlong, and then the tenth, hundredth, and thousandth parts of a furlong, which the tortoise has successively advanced before him, were equal; but if we suppose that he has walked the first furlong in 10 minutes, he will require only one minute to walk the tenth of a furlong, and $\frac{1}{10}$ of a minute to walk the hundredth, &c. The intervals of time therefore, which Achilles will require to pass over the space gained by the tortoise, during the preceding time, will go on decreasing in the following manner: 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c; and this series forms a sub-decuple geometrical progression, the sum of which is equal to $11\frac{1}{9}$, or the interval of time at the end of which Achilles will have reached the tortoise.

PROBLEM II.

If the hour and minute hands of a clock both begin to move exactly at noon; at what points of the dial-plate will they be successively in conjunction, during a whole revolution of the twelve hours?

This problem, considered in a certain manner, is in nothing different from the preceding. The minute hand acts here the part which Achilles did in the former, and the hour hand, which moves twelve times slower, that of the tortoise. In the last place, if we suppose the hour hand to be beginning a second revolution, and the minute hand to be beginning a first, the distance which the one has gained over the other will be a whole revolution of the dial-plate. When the minute hand has made one revolution, the hour hand will have made but one twelfth of a revolution, and so on progressively. To resolve the problem therefore, we need only apply to these data, the method employed in the former case, and we shall find, that the interval from noon, to the point where the two hands come again into conjunction, will be $\frac{1}{11}$ of a whole revolution, or, what amounts to the same thing, one hour and $\frac{1}{11}$ of an hour. They will afterwards be in conjunction at 2 hours and $\frac{2}{11}$, 3 hours and $\frac{3}{11}$, 4 hours and $\frac{4}{11}$, &c; and, lastly, at 11 hours $\frac{11}{11}$, that is to say at 12 hours.

PROBLEM III.

A courtier having performed some very important service to his sovereign, the latter, wishing to confer on him a suitable reward, desired him to ask whatever he thought proper, promising that it should be granted. The courtier, who was well acquainted with the science of numbers, only requested that the monarch would give him a quantity of wheat equal to that which would arise from one grain doubled sixty-three times successively. What was the value of the reward?

The origin of this problem is related in so curious a manner by Al-Sephadi, an Arabian author, that it deserves to be mentioned. A mathematician named Sessa, says he, the son of Daher, the subject of an Indian prince, having invented the game of chess, his sovereign was highly pleased with the invention, and wishing to confer on him some reward worthy of his magnificence, desired him to ask whatever he thought proper, assuring him that it should be granted. The mathematician, however, only asked a grain of wheat for the first square of the chess-board, two for the second, four for the third, and so on to the last or sixty-fourth. The prince at first was almost incensed at this demand, conceiving that it was ill suited to his liberality, and ordered his vizier to comply with Sessa's request; but the minister was much astonished when, having caused the quantity of corn necessary to fulfil the prince's order to be calculated, he found that all the grain in the royal granaries, and that even of all his subjects, and in all Asia, would not be sufficient. He therefore informed the prince, who sent for the mathematician, and candidly acknowledged that he was not rich enough to be able to comply with his demand, the ingenuity of which astonished him still more than the game he had invented.

Such is then the origin of the game of chess, at least according to the Arabian historian Al-Sephadi. But it is not our business here to discuss the truth of this story; our business being to calculate the number of grains demanded by the mathematician Sessa.

It will be found by calculation, that the 64th term of the double progression, beginning with unity, is 9223372036854775808. But the sum of all the terms of a double progression, beginning with unity, may be obtained by doubling the last term and subtracting from it unity. The number therefore of the grains of wheat equal to Sessa's demand, will be 18446744073709551615. Now, if a standard pint contains 9216 grains of wheat, a gallon will contain 73728, and, as eight gallons make one bushel, if we divide the above result by eight times 73728, we shall have 31274997412295 for the number of the bushels of wheat necessary to discharge the promise of the Indian king; and if we suppose that one acre of land is capable of producing in one year, thirty bushels of wheat, to produce this quantity would require 1042499913743 acres, which make more than eight times the surface of the whole globe; for the diameter of the earth being supposed equal to 7930 miles, its whole surface, comprehending land and water, will amount to very little more than 126437889177 square acres.

Dr. Wallis considers the matter in a manner somewhat different, and says, in his Arithmetic, that the quantity of wheat necessary to discharge the promise made to Sessa, would form a pyramid nine miles English in length, breadth and height; which is equal to a parallelopiped mass having nine square leagues for its base, and of the uniform height of one league. But as one league contains 15840 feet, this solid would be equivalent to another one foot in height and having a base equal to 142560 square leagues. Hence it follows, that the above quantity of wheat would cover, to the height of one foot, 142560 square leagues; an extent of surface equal to eleven times that of Britain, which when every reduction is made will be found to contain little more than 12674 square leagues.

If the price of a bushel of wheat be estimated at ten shillings, the value of the above quantity will amount to 15637498706147*l.* 10*s.* a sum which, in all probability, far surpasses all the riches on the earth.

Another problem of the same kind is proposed in the following manner:

A gentleman taking a fancy to a horse, which a horse-dealer wished to dispose of at as high a price as he could, the latter, to induce the gentleman to become a purchaser, offered to let him have the horse for the value of the twenty-fourth nail in his shoes, reckoning one farthing for the first nail, two for the second, four for the third, and so on to the twenty-fourth. The gentleman, thinking he should have a good bargain, accepted the offer: what was the price of the horse?

By calculating as before, the 24th term of the progression 1, 2, 4, 8, &c, will be found to be 8388608, equal to the number of farthings the purchaser ought to give for the horse. The price therefore amounted to 8738*l.* 2*s.* 8*d*, which is more than any Arabian horse, even of the noblest breed, was ever sold for.

Had the price of the horse been the value of all the nails, at a farthing for the first, two for the second, four for the third, and so on, the sum would have been double the above number, minus the first term, or 16777215 farthings, that is 17476*l.* 5*s.* 3¾*d.*

We shall conclude this chapter with some physico-mathematical observations on the prodigious fecundity, and the progressive multiplication, of animals and vegetables, which would take place, if the powers of nature were not continually meeting with obstacles.

I. It is not astonishing that the race of Abraham, after sojourning 260 years in Egypt, should have formed a nation capable of giving uneasiness to the sovereigns of that country. We are told in the sacred writings, that Jacob settled in Egypt with 70 persons; now if we suppose that among these 70 persons, there were twenty too far advanced in life, or too young, to have children; that, of the remaining 50, 25 were males and as many females, forming 25 married couples, and that each couple, in the space of 25 years produced, one with another, 8 children, which will not appear incredible in a country celebrated for the fecundity of its inhabitants, we shall find that, at the end of 25 years, the above 70 persons may have increased to 270; from which if we deduct those who died, there will perhaps be no exaggeration in making them amount to 210. The race of Jacob therefore, after sojourning 25 years in Egypt, may have been tripled. In like manner, these 210 persons, after 25 years more, may have increased to 630; and so on in triple geometrical progression: hence it follows that, at the end of 225 years, the population may have amounted to 1377810 persons, among whom there might easily be five or six hundred thousand adults fit to bear arms.

II. If we suppose that the race of the first man, making a proper deduction for those who died, may have been doubled every twenty years, which certainly is not inconsistent with the powers of nature, the number of men, at the end of five centuries, may have amounted to 1048576. Now, as Adam lived about 900 years, he may have seen therefore, when in the prime of life, that is to say about the five hundredth year of his age, a posterity of 1048576 persons.

III. How great would be the multiplication of many animals, did not the difficulty of finding food, the continual war which they carry on against each other, or the numbers of them consumed by man, set bounds to their propagation! It might easily be proved, that the breed of a sow, which brings forth six young, two males and four females, if we suppose that each female produces every year afterwards six young, four of them females and two males, would in twelve years amount to 33554230.

Several other animals, such as rabbits and cats, which go with young only for a few weeks, would multiply with still greater rapidity: in half a century the whole earth would not be sufficient to supply them with food, nor even to contain them!

If all the ova of a herring were fecundated, a very few years would be sufficient to make its posterity fill the whole ocean; for every oviparous fish contains thousands of ova, which it deposits in spawning time. Let us suppose that the number of ova amounts only to 2000, and that these produce as many fish, half males and half females; in the second year there would be more than 200000; in the third, more than 200000000; and in the eighth year the number would exceed that expressed by 2 followed by twenty-four ciphers. As the earth contains scarcely so many cubic inches, the ocean, if it occupied the whole globe, would not be sufficient to contain all these fish, the produce of one herring in eight years!

IV. Many vegetable productions, if all their seeds were put into the earth, would in a few years cover the whole surface of the globe. The hyosciamus, which of all the known plants produces perhaps the greatest number of seeds, would for this purpose require no more than four years. According to some experiments, it has been found that one stem of the hyosciamus produces sometimes more than 50000 seeds: now if we admit the number to be only

10000, at the fourth crop it would amount to a 1 followed by sixteen ciphers. But as the whole surface of the earth contains no more than 5507634452576256 square feet; if we allow to each plant only one square foot, it will be seen that the whole surface of the earth would not be sufficient for the plants produced from one hyosciamus at the end of the fourth year!

§ III.

Of some other Progressions, and particularly Harmonical Progression.

THREE numbers are in harmonical proportion, when the first is to the last, as the difference between the first and the second, is to that between the second and the third. Thus, the numbers 6, 3, 2, are in harmonical proportion; for 6 is to 2, as 3, the difference between the two first numbers, is to 1, the difference between the two last. This kind of relation is called harmonical, for a reason which will be seen hereafter.

I. Two numbers being given, a third which shall form with them harmonical proportion may be found, by multiplying these two numbers, and dividing their product by the excess of the double of the first over the second Thus, if 6 and 3 be given, we must multiply 6 by 3, and divide the product 18 by 9, which is the excess of 12 the double of 6 over 3, the second of the numbers given. In this case the quotient will be 2.

It may hence be readily seen that, in one sense, it is not always possible to find a third number in harmonical proportion with two others; for when the first is less, if its double be equal to or less than the second, the result will be an infinite or a negative number. Thus, the third harmonic proportional to 2 and 4 is infinite; for it will be found that the number sought is equal to 8 divided by $4 - 4$, or 0. But every person, in the least acquainted with arithmetic, knows that the more the denominator of a fraction is inferior to unity, the greater the fraction; consequently, a fraction which has 0 for its denominator is infinite.

If the double of the first number be less than the second, as would be the case were it proposed to find a third harmonical to 2 and 6, the required divisor will be a negative number. Thus, in the proposed example of 2 and 6, it will be -2; and therefore the third harmonical required will be 12 divided by -2, that is -6 *.

But this inconvenience, if it be one, is not to be apprehended when the greater number is the first term of the proportion; for if the first exceeds the second, much more will its double exceed it. In this case therefore, the third harmonical will always be a finite and positive number.

II. When three numbers, in decreasing harmonical proportion, are given, for example 6, 3, 2, it is easy to find a fourth: nothing is necessary but to find a third harmonical to the two last, and this will be the fourth. The third and fourth may, in like manner, be employed to find a fifth, and so on; and this will form what is called an harmonical progression, which may be always continued decreasing. In the present example, this series will be found to be $6, 3, 2, \frac{6}{4}, \frac{6}{5}, \frac{6}{6}, \frac{6}{7}, \frac{6}{8}$, &c.

or $6, 3, 2, \frac{3}{2}, \frac{6}{5}, 1, \frac{6}{7}, \frac{3}{4}$, &c.

Had the two first numbers been 2 and 1, we should have had the harmonical progression

$$2, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \text{\&c.}$$

It is a remarkable property, therefore, of the series of

* See what has been already said in regard to negative quantities, in the article on arithmetical progression.

fractions, having for their numerators unity, and for their denominators the numbers of the natural progression, that they are in harmonical progression.

Besides the numerical relation already mentioned, we find indeed, in the series of these numbers, all the musical concords possible; for the ratio of 1 to $\frac{1}{2}$ gives the octave; that of $\frac{1}{2}$ to $\frac{1}{3}$, or of 3 to 2, the fifth; that of $\frac{1}{3}$ to $\frac{1}{4}$, or of 4 to 3, the fourth; that of $\frac{1}{4}$ to $\frac{1}{5}$, or of 5 to 4, the third major; that of $\frac{1}{5}$ to $\frac{1}{6}$, or of 6 to 5, the third minor; that of $\frac{1}{8}$ to $\frac{1}{9}$, or of 9 to 8, the tone major, and that of $\frac{1}{9}$ to $\frac{1}{10}$, or of 10 to 9, the tone minor. But this will be explained at more length in that part of this work which treats of music.

It is a curious property, that the three mean proportionals, viz, the arithmetical mean, the geometrical mean, and the harmonical mean, between any two numbers, are always in continued geometrical progression. So, between the two numbers 8 and 2, the arithmetical mean is 5, the geometrical mean is 4, and the harmonical mean is $3\frac{1}{5}$; and it is obvious that $5 : 4 :: 4 : 3\frac{1}{5}$.

PROBLEM.

What is the Sum of the Infinite Series of Numbers in Harmonical Progression, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, *&c.?*

It has been already seen, that a series of numbers in geometrical progression, if continued *in infinitum*, will always be equal to a finite number, which may easily be determined. But is the case the same in the present problem?

We will venture to reply in the negative, though an author, in the *Journal de Trevoux*, has bestowed great labour in endeavouring to prove that the sum of these fractions is finite. But his reasoning consists of mere paralogisms, which he would not have employed had he been more of a mathematician; for it can be demonstrated that the series 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c, may be always continued in such a manner as to exceed any finite number whatever.

§ IV.

Of various Progressions Decreasing in infinitum, the Sums of which are known.

I. A VARIETY of decreasing progressions, which have served to exercise the ingenuity of mathematicians, may be formed according to different laws. Thus, for example, the numerator being constantly unity, the denominators may increase in the ratio of the triangular numbers 1, 3, 6, 10, 15, 21, &c. Of this kind is the following progression: $\frac{1}{1}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{15}$, $\frac{1}{21}$, &c. Its sum is finite, and exactly equal to 2, or $1\frac{1}{1}$.

In like manner the sum of a progression having unity constantly for its numerators, and the pyramidal numbers for its denominators, as, 1, $\frac{1}{4}$, $\frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{35}$, $\frac{1}{56}$, &c, is equal to $1\frac{1}{2}$.

That where the denominators are the pyramidals of the second order, as 1, $\frac{1}{5}$, $\frac{1}{15}$, $\frac{1}{35}$, $\frac{1}{70}$, $\frac{1}{126}$, &c, is equal to $1\frac{1}{3}$.

That where they are pyramidals of the third order, as 1, $\frac{1}{6}$, $\frac{1}{21}$, $\frac{1}{56}$, $\frac{1}{126}$, $\frac{1}{252}$, &c, is equal to $1\frac{1}{4}$.

The law therefore which these sums follow, is evident: and if the sum of a similar progression, that, for example, where the denominators are the pyramidals of the tenth order, were required, we might easily reply that it is equal to $1\frac{1}{11}$.

II. Let us now assume the following progression, 1, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, $\frac{1}{25}$, $\frac{1}{36}$, &c, in which the denominators are the squares of the numbers of the natural progression.

If the reader is desirous to know its sum, we shall observe, with Mr. John Bernoulli, by whom it was first found, that it is finite, and equal to the square of the circumference of the circle divided by 6, or $\frac{1}{6}$ of $3{\cdot}14159^2$.

As to that in which the denominators are the cubes of the natural numbers, Mr. Bernoulli acknowledges that he had not been able to discover it.

Those who are fond of researches of this kind, may consult a work of James Bernoulli, entitled *Tractatus de Seriebus Infinitis*, which is at the end of another published at Bâle in 1713, under the title of *Ars Conjectandi*, where they will find ample satisfaction. They may also consult various other memoirs both of John Bernoulli, to be found in the collection of his works, and of Euler, published in the Transactions of the Imperial Academy of Sciences at Petersburg.

CHAPTER VIII.

Of Combinations and Permutations.

BEFORE we enter on the present subject, it will be necessary to explain the method of constructing a sort of table, invented by Pascal *, called the Arithmetical triangle, which is of great utility for shortening calculations of this kind.

First, form a band AB of ten equal squares, and below it another CD of the like kind, but shorter by one square on the left, so that it shall contain only nine squares; and continue in this manner always making each

* This is a mistake in Montucla, as the triangle was invented some ages before Pascal; see Dr. Hutton's Tracts, vol. 1, pa. 231.

successive band a square shorter. We shall thus have a series of squares disposed in vertical and horizontal bands, and terminating at each end in a single square, so as to form a triangle, on which account this table has been called the arithmetical triangle. The numbers with which it is to be filled up, must be disposed in the following manner.

A	1	1	1	1	1	1	1	1	1	1	B
	C	1	2	3	4	5	6	7	8	9	D
			1	3	6	10	15	21	28	36	
				1	4	10	20	35	56	84	
					1	5	15	35	70	126	
						1	6	21	56	126	
							1	7	28	84	
								1	8	36	
									1	9	
										1	
										E	

In each of the squares of the first band AB, inscribe unity, as well as in each of those on the diagonal AE.

Then add the number in the first square of the band CD, which is unity, to that in the square immediately above it, and write down the sum 2, in the following square. Add this number, in the like manner, to that in the square above it, which will give 3, and write it down in the next square. By these means we shall have the series of the natural numbers, 1, 2, 3, 4, 5, &c. The same method must be followed to fill up the other horizontal bands; that is to say, each square ought always to contain the sum of the number in the preceding square of the same row, and that which is immediately above it. Thus, the number 15, which occupies the fifth square of the third band, is equal to the sum of 10 which stands in the preceding square, and of 5 which is in the square above it.

The case is the same with 21, which is the sum of 15 and 6; with 35 in the fourth band, which is the sum of 15 and 20; &c.

The first property of this table is, that it contains, in its horizontal bands, the natural, triangular, pyramidal, &c, numbers; for in the second, we have the natural numbers 1, 2, 3, 4, &c; in the third, the triangular numbers, 1, 3, 6, 10, 15, &c; in the fourth, the pyramidals of the first order, 1, 4, 10, 20, 35, &c; in the fifth, the pyramidals of the second order 1, 5, 15, 35, 70, &c. This is a necessary consequence of the manner in which the table is formed; for it may be readily perceived, that the number in each square is always the sum of those which fill the preceding squares on the left, in the band immediately above.

The same numbers will be found in the bands parallel to the diagonal, or the hypothenuse of the triangle.

But a property still more remarkable, which can be comprehended only by such of our readers as are acquainted with algebra, is, that the perpendicular bands exhibit the co-efficients belonging to the different members of any power to which a binomial, as $a+b$, can be raised. The third band contains those of the three members of the square; the fourth those of the four members of the cube; the fifth, those of the five members of the biquadrate. But, without enlarging farther on this subject, we shall proceed to explain what is meant by combinations.

By combinations are understood the various ways that different things, the number of which is known, can be chosen or selected, taking them one by one, two by two, three by three, &c, without regard to their order. Thus, for example, if it were required to know in how many different ways the four letters *a*, *b*, *c*, *d*, could be arranged, taking them two and two, it may be readily seen that we can form with them the following combinations, *ab*, *ac*, *ad*, *bc*, *bd*, *cd*: four things, therefore, may be combined, two and two, six different ways. Three of these letters may be combined four ways, *abc*, *abd*, *acd*, *bcd*; hence the combinations of four things, taken three and three, are only four.

In combinations, properly so called, no attention is paid to the order of the things; and for this reason we have made no mention of the following combinations, *ba*, *ca*, *da*, *cb*, *db*, *dc*. If, for example, four tickets, marked *a*, *b*, *c*, *d*, were put into a hat, and any one should bet to draw out the tickets *a* and *d*, either by taking two at one time, or taking one after another, it would be of no importance whether *a* should be drawn first or last: the combinations *ad* or *da* ought therefore to be here considered only as one.

But if any one should bet to draw out *a* the first time, and *d* the second, the case would be very different; and it would be necessary to attend to the order in which these four letters may be taken and arranged together, two and two: it may be easily seen that the different ways are, *ab*, *ba*, *ac*, *ca*, *ad*, *da*, *bc*, *cb*, *bd*, *db*, *cd*, *dc*. In like manner, these four letters might be combined and arranged, three and three, 24 ways, as *abc*, *acb*, *bac*, *bca*, *cab*, *cba*, *adb*, *abd*, *dba*, *dab*, *bad*, *bda*, *acd*, *adc*, *dac*, *dca*, *cad*, *cda*, *bcd*, *dbc*, *cbd*, *bdc*, *cbd*, *dcb*. This is what is called permutation and change of order.

PROBLEM I.

Any number of things whatever being given; to determine in how many ways they may be combined two and two, three and three, &c, without regard to order.

This problem may be easily solved by making use of the arithmetical triangle. Thus, for example, if there are eight things to be combined, three and three, we must take the ninth vertical band, or, in all cases, that band the order of which is expressed by a number exceeding by unity the number of things to be combined; then the fourth horizontal band, or that the order of which is greater by unity than the number of the things to be taken together; then in the common square of both will be found the number of the combinations required, which in the present example will be 56.

But as an arithmetical triangle may not always be at hand, or as the number of things to be combined may be too great to be found in such a table, the following simple method may be employed.

The number of the things to be combined, and the manner in which they are to be taken, viz. two and two, or three and three, &c, being given:

1st, Form two arithmetical progressions, one in which the terms go on decreasing by unity, beginning with the given number of things to be combined; and the other consisting of the series of the natural numbers 1, 2, 3, 4, &c.

2d, Then take from each as many terms as there are things to be arranged together in the proposed combination.

3d, Multiply together the terms of the first progression, and do the same with those of the second.

4th, In the last place, divide the first product by the second, and the quotient will be the number of the combinations required.

§ I.

In how many ways can 90 things be combined, taking them two and two?

According to the above rule we must multiply 90 by 89, and divide the product 8010 by the product of 1 and

2, that is 2: the quotient 4005 will be the number of the combinations resulting from 90 things, taken two and two.

Should it be required, in how many ways the same things can be combined three and three, the problem may be answered with equal ease; for we have only to multiply together 90, 89, 88, and to divide the product 704880 by that of the three numbers 1, 2, 3; the quotient 117480 will be the number required.

In like manner, it will be found that 90 things may be combined by four and four, 2555190 ways; for if the product of 90, 89, 88 and 87 be divided by 24, the product of 1, 2, 3, 4, we shall have the above result.

In the last place, if it be required, what number of combinations the same 90 things, taken five and five, are susceptible of, it will be found, by following the rule, that the answer is 43949268.

§ II.

Were it asked, how many conjunctions the seven planets could form with each other, two and two, we might reply 21; for, according to the general rule, if we multiply 7 by 6, which will give 42, and divide that number by the product of 1 and 2, that is 2, the quotient will be 21.

If we wished to know the number of all the conjunctions possible of these seven planets, two and two, three and three, &c; by finding separately the number of the conjunctions two and two, then those of three and three, &c, and adding them together, it will be seen that they amount to 120.

The same result might be obtained by adding the seven terms of the double geometrical progression 1, 2, 4, 8, 16, 32, 64, which will give 127. But from this number we must deduct 7, because when we speak of the conjunction of a planet, it is evident that two of them, at least, must be united; and the number 127 comprehends all the ways in which seven things can be taken one and one, two and two, three and three, &c. In the present example therefore, we must deduct the number of the things taken one and one, for a single planet cannot form a conjunction.

PROBLEM II.

Any number of things being given; to find in how many ways they can be arranged.

This problem may be easily solved by following the method of induction; for

1st, One thing *a* can be arranged only in one way: in this case therefore the number of arrangements is = 1.

2d, Two things may be arranged together two ways; for with the letters *a* and *b* we can form the arrangements *ab* and *ba:* the number of arrangements therefore is equal to 2, or the product of 1 and 2.

3d, The arrangements of three things *a*, *b*, *c*, are in number six; for *ab* can form with *c*, the third, three different ones, *bac*, *bca*, *cba*, and there can be no more. Hence it is evident that the required number is equal to the preceding multiplied by 3, or to the product of 1, 2, 3.

4th, If we add a fourth thing, for instance *d*, it is evident that, as each of the preceding arrangements may be combined with this fourth thing four ways, the above number 6 must be multiplied by 4 to obtain that of the arrangements resulting from four things; that is to say, the number will be 24, or the product of 1, 2, 3, 4.

It is needless to enlarge farther on this subject; for it may be easily seen that, whatever be the number of the things given, the number of the arrangements they are susceptible of may be found, by multiplying together as many terms of the natural arithmetical progression as there are things proposed.

REMARK.

1st, It may sometimes happen that, of the things proposed, one of them is repeated, as *a*, *a*, *b*, *c*. In this case, where two of the four things proposed are the same, it will be found that they are susceptible only of 12 arrangements, instead of 24; and that five, where two are the same, can form only 60, instead of 120.

But if three, of four things, were the same, there would be only 4 combinations instead of 24; and five things, if three of them were the same, would give only 20, instead of 120, or a sixth part. But as the arrangements of which two things are susceptible amount to 2, and as those which can be formed with three things are 6, we may thence deduce the following rule:

In any number of things, of which the different arrangements are required, if one of them be several times repeated, divide the number of arrangements, found according to the general rule, by that of the arrangements which would be given by the things repeated, if they were different, and the quotient will be the number required.

2d, In the number of things, the different arrangements of which are required, if there are several of them which occur several times, one twice for example, and another three times; nothing will be necessary, but to find the number of the arrangements according to the general rule, and then to divide it by the product of the numbers expressing the arrangements which each of the things repeated would be susceptible of, if instead of being the same, they were different. Thus, in the present case, as the things which occur twice, would be susceptible of two arrangements, if they were different; and as those which occur thrice would, under the like circumstances, give six; we must multiply 6 by 2, and the product 12 will be the number, by which that found according to the general rule, ought to be divided. Thus, for example, the five letters *a, a, b, b, b,* can be arranged only 10 different ways: for, if they were different, they would give 120 arrangements; but as one of them occurs twice, and another thrice, 120 must be divided by the product of 2 and 6, or 12, which will give 10.

By observing the precepts given for the solution of this problem, the following questions may be resolved.

§ I.

A club of seven persons agreed to dine together, every day successively, as long as they could sit down to table differently arranged. How many dinners would be necessary for that purpose?

It may be easily found that the required number is 5040, which would require 13 years and more than 9 months.

§ II.

The different anagrams which can be formed with any word, may be found in like manner. Thus, for example, if it be required, how many different words can be formed with the four letters of the word AMOR, which will give all the possible anagrams of it, we shall find that they amount to 24, or the continued product of 1, 2, 3, 4. We shall here give them in their regular order.

AMOR	MORA	ORAM	RAMO
AMRO	MOAR	ORMA	RAOM
AOMR	MROA	OARM	RMAO
AORM	MRAO	OAMR	RMOA
ARMO	MAOR	OMRA	ROAM
AROM	MARO	OMAR	ROMA

Hence it appears that the Latin anagrams of the word *amor*, are in number seven, viz, *Roma, mora, maro, oram,* tour which he made to the canal of Orleans, some square porcelain tiles, divided by a diagonal into two triangles of different colours, destined for paving a chapel and some apartments, he was induced to try in how many different ways they could be joined side by side, in order to form different figures. In the first place, it may be readily seen that a single square (Plate II.) according to its position can form four different figures, which however may be reduced to two, as there is no other difference between the first and the third, or between the second and the fourth, than what arises from the transposition of the shaded triangle into the place of the white one.

Now, if two of these squares be combined together, the result will be 64 different ways of arrangement; for, in that of two squares, one of them may be made to assume four different situations, in each of which the other may be changed 16 times. The result therefore will be 64 combinations, as may be seen in the plate.

We must however observe, with Father Sebastian, that one half of these combinations are only a repetition of the other, in a contrary direction, which reduces them to 32; and if attention were not paid to situation, they might be reduced to 10.

In like manner, we might combine three, four, five, &c, squares together, and in that case it would be found, that three squares are capable of forming 128 figures; that four could form 256, &c.

The immense variety of compartments which arise, in this manner, from so small a number of elements is really astonishing. Father Sebastian gives thirty different kinds, selected from a hundred; and these even are only a very small part of those which might be formed. Some of the most remarkable of them are exhibited in the 2d plate.

In consequence of Father Sebastian's memoir, Father Douat, one of his associates, was induced to pursue this subject still farther, and to publish, in the year 1722, a large work*, in which it is considered in a different manner. In this work it may be seen that four squares, each divided into two triangles of different colours, repeated and changed in every manner possible, are capable of forming 256 different figures; and that these figures themselves, taken two and two, three and three, and so on, will form a prodigious multitude of compartments, engravings of which occupy the greater part of the book.

It is rather surprising that this idea should have been so little employed in architecture; as it might furnish an inexhaustible source of variety in pavements, and other works of the like kind. However this may be, it forms the object of a pastime, called by the French *Jeu du Parquet*. The instrument employed for this pastime, consists of a small table, having a border round it, and capable of receiving 64 or a hundred small squares, each divided into two triangles of different colours, with which people amuse themselves in endeavouring to form agreeable combinations.

CHAPTER IX.

Application of the Doctrine of Combinations to Games of Chance and to Probabilities.

THOUGH nothing, on the first view, seems more foreign to the province of the mathematics than chance, the powers of analysis have, as we may say, enchained this Proteus, and subjected it to calculation. It has found means to measure the different degrees of probability;

* It is entitled *Methode pour faire une infinité de dessins différents, avec des carreaux mi-partis de deux couleurs par une ligne diagonale; ou, Observations du P. D. Douat, religieux Carme de la P. de T. sur un Memoire inseré dans l'Hist. de l'Acad. royale des Sciences de Paris, année* 1704, *par le P. S. Truchet, religieux du même ordre, Paris* 1722, in 4to.

and this has given birth to a curious branch of the mathematics, the principles of which we shall here explain.

When an event can take place different ways, it is evident that the probability of its happening in a certain determinate manner, will be greater when, of the whole of the ways in which it can happen, the greater number determine it to happen in that manner. In a lottery, for example, every one knows that the probability or hope of obtaining a prize, is greater according as the number of prizes is greater, and as the total number of the tickets is less. The probability, therefore, of an event, is in the compound ratio of the number of the cases which can produce it, taken directly, and of the total number of those according to which it may be varied, taken inversely; consequently it may be expressed by a fraction, having for its numerator the number of the favourable cases, and for its denominator the whole of the cases.

Thus, in a lottery consisting of a thousand tickets, 25 of which only are prizes, the chance of obtaining one of the latter will be represented by $\frac{25}{1000}$, or $\frac{1}{40}$; if the number of the prizes were 50, this probability would be double, for in that case it would be equal to $\frac{1}{20}$; but, on the other hand, if the whole number of tickets, instead of a thousand, were two thousand, the probability would be only one half of the former, that is $\frac{1}{80}$. If the whole number of tickets were infinitely great, the number of prizes still remaining the same, the probability would be infinitely small; and if the whole number of tickets were prizes, it would become certainty, and in that case would be expressed by unity.

Another principle of this theory, necessary to be here explained, the enunciation of which will be sufficient to show the truth of it, is as follows:

We play an equal game, when the money deposited is in direct proportion to the probability of gaining the stake;

for, to play an equal game, is nothing else than to deposit a sum so proportioned to the probability of winning, that, after a great number of throws or games, the player may find himself nearly at par; but for this purpose, the sums deposited must be proportioned to the degree of probability, which each of the players has in his favour. Let us suppose, for example, that A bets against B on a throw of the dice, and that the chances are two to one in favour of A; the game will be equal if, after a great number of throws, the parties separate nearly without any loss; but as there are two chances in favour of A, and only one in favour of B, after three hundred throws A will have gained nearly two hundred, and B one hundred; A therefore ought to deposit 2 and B only one, for by these means, as A in winning two hundred throws will gain 200, B in winning a hundred throws will gain 200 also. In such cases therefore, it is said that two to one may be betted in favour of A.

PROBLEM I.

In tossing up, what probability is there of throwing a head several times successively, or a tail; or, in playing with several pieces, what probability is there that they will be all heads, or all tails?

In this game, which is well known, it is evident, 1st, That as there is no reason why a head should come up rather than a tail, or a tail rather than a head, the probability that one of the two will be the case is equal to $\frac{1}{2}$, or an equal bet may be taken, for or against.

But if the game were for two throws, and any one should bet, that a head will come up twice, it must be observed, that all the combinations of head or tail, which can take place in two successive throws with the same piece, are *head, head; head, tail; tail, head; tail, tail;* one of which only gives head, head. There is therefore only one case in 4 which can make the person win who bets to throw a head twice in succession; consequently the probability of this event is only $\frac{1}{4}$; and he who bets in favour of two heads, ought to deposit a crown, and the person who bets against him ought to deposit three; for the latter has three chances of winning, while the former has only one. To play an equal game then, the sums deposited by each, ought to be in this proportion.

It will be found also, that he who bets to throw a *head* three times in succession, will have in his favour only one of the eight combinations of head and tail, which may result from three throws of the same piece. The probability of this event therefore, is $\frac{1}{8}$, while that in favour of his adversary will be $\frac{7}{8}$. Consequently, to play an equal game, he ought to stake 1 against 7.

It is needless to go over all the other cases; for it may be easily seen, that the probability of throwing a *head* four times successively, is $\frac{1}{16}$; five times successively, $\frac{1}{32}$, &c.

It is unnecessary also to enumerate all the different combinations which may result from *head* or *tail;* but in regard to probabilities, the following simple rule may be employed.

The probabilities of two or more single events being known, the probability of their taking place all together may be found, by multiplying together the probabilities of these events, considered singly.

Thus the probability of throwing a head, considered singly, being expressed at each throw by $\frac{1}{2}$, that of throwing it twice in succession, will be $\frac{1}{2} \times \frac{1}{2}$ or $\frac{1}{4}$; that of throwing it three times, in three successive throws, will be $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$, or $\frac{1}{8}$, &c.

2d. The problem, to determine the probability of throwing, with two, three, or four pieces, all *heads* or all *tails*, may be resolved by the same means. When two pieces are tossed up, there are four combinations of *head* and *tail*, one of which only is all heads. When three pieces are tossed up together, there are 8, one of which only gives all *heads*, &c. The probabilities of these cases therefore, are the same as those of the cases similar to them, which we have already examined.

It may be easily seen indeed, without the help of analysis, that these two questions are absolutely the same; and the following mode of reasoning may be employed to prove it. To toss up the two pieces A and B together, or to toss them up in succession, giving time to A, the first, to settle before the other is tossed up, is certainly the same thing. Let us suppose then, that when A, the first, has settled, instead of tossing up B, the second, A, the first, is taken from the ground, in order to be tossed up a second time; this will be the same thing as if the piece B had been employed for a second toss; for by the supposition they are both equal and similar, at least in regard to the chance whether head or tail will come up. Consequently, to toss up the two pieces A and B, or to toss up twice in succession the piece A, is the same thing. Therefore, &c.

3d. We shall now propose the following question: What may a person bet, that, in two throws, a *head* will come up at least once? By the above method it will be found, that the chances are 3 to 1. In two throws, indeed, there are four combinations, three of which give at least a head once in the two throws, and one only which gives all tails; hence it follows, that there are three combinations in favour of the person who bets to bring a head once in two throws, and only one against him.

PROBLEM II.

Any number of dice being given; to determine what probability there is of throwing an assigned number of points.

We shall first suppose that the dice are of the ordinary

REMARK.

1st, It may sometimes happen that, of the things proposed, one of them is repeated, as *a*, *a*, *b*, *c*. In this case, where two of the four things proposed are the same, it will be found that they are susceptible only of 12 arrangements, instead of 24; and that five, where two are the same, can form only 60, instead of 120.

But if three, of four things, were the same, there would be only 4 combinations instead of 24; and five things, if three of them were the same, would give only 20, instead of 120, or a sixth part. But as the arrangements of which two things are susceptible amount to 2, and as those which can be formed with three things are 6, we may thence deduce the following rule:

In any number of things, of which the different arrangements are required, if one of them be several times repeated, divide the number of arrangements, found according to the general rule, by that of the arrangements which would be given by the things repeated, if they were different, and the quotient will be the number required.

2d, In the number of things, the different arrangements of which are required, if there are several of them which occur several times, one twice for example, and another three times; nothing will be necessary, but to find the number of the arrangements according to the general rule, and then to divide it by the product of the numbers expressing the arrangements which each of the things repeated would be susceptible of, if instead of being the same, they were different. Thus, in the present case, as the things which occur twice, would be susceptible of two arrangements, if they were different; and as those which occur thrice would, under the like circumstances, give six; we must multiply 6 by 2, and the product 12 will be the number, by which that found according to the general rule, ought to be divided. Thus, for example, the five letters *a*, *a*, *b*, *b*, *b*, can be arranged only 10 different ways: for, if they were different, they would give 120 arrangements; but as one of them occurs twice, and another thrice, 120 must be divided by the product of 2 and 6, or 12, which will give 10.

By observing the precepts given for the solution of this problem, the following questions may be resolved.

§ I.

A club of seven persons agreed to dine together, every day successively, as long as they could sit down to table differently arranged. How many dinners would be necessary for that purpose?

It may be easily found that the required number is 5040, which would require 13 years and more than 9 months.

§ II.

The different anagrams which can be formed with any word, may be found in like manner. Thus, for example, if it be required, how many different words can be formed with the four letters of the word AMOR, which will give all the possible anagrams of it, we shall find that they amount to 24, or the continued product of 1, 2, 3, 4. We shall here give them in their regular order.

AMOR	MORA	ORAM	RAMO
AMRO	MOAR	ORMA	RAOM
AOMR	MROA	OARM	RMAO
AORM	MRAO	OAMR	RMOA
ARMO	MAOR	OMRA	ROAM
AROM	MARO	OMAR	ROMA

Hence it appears that the Latin anagrams of the word *amor*, are in number seven, viz, *Roma*, *mora*, *maro*, *oram*, *ramo, armo, orma.* But in the proposed word, if one or more letters were repeated, it would be necessary to follow the precepts already given. Thus, the word *Leopoldus*, where the letter *l* and the letter *o* both occur twice, is susceptible of only 90720 different arrangements, or anagrams, instead of 362880, which it would form, if none of the letters were repeated; for, according to the before-mentioned rule, we must divide this number by the product of 2 by 2, or 4, which will give 90720.

The word *studiosus*, where the *u* occurs twice and the *s* thrice, is susceptible of only 30240 arrangements; for the arrangements of the 9 letters it contains, which are in number 362880, must be divided by the product of 2 and 6, or 12, and the quotient will be 30240.

In this manner may be found the number of all the possible anagrams of any word whatever; but it must be observed that however few be the letters of which a word is composed, the number of the arrangements thence resulting will be so great as to require considerable labour to find them.

§ III.

How many ways can the following verse be varied, without destroying the measure:

Tot tibi sunt dotes, Virgo, quot sidera cælo?

THIS verse, the production of a devout Jesuit of Louvain, named Father Bauhuys, is celebrated on account of the great number of arrangements of which it is susceptible, without the laws of quantity being violated; and various mathematicians have exercised or amused themselves with finding out the number. Erycius Puteanus took the trouble to give an enumeration of them in forty-eight pages, making them amount to 1022, or the number of the stars comprehended in the catalogues of the ancient astronomers; and he very devoutly observes, that the arrangements of these words, as much exceed the above number, as the perfections of the Virgin exceed that of the stars*.

Father Prestet, in the first edition of his Elements of the Mathematics, says that this verse is susceptible of 2196 variations; but in the second edition he extends the number to 3276.

Dr. Wallis in the edition of his Algebra printed at Oxford, in 1693, makes them amount to 3096.

But none of them has exactly hit the truth, as has been remarked by James Bernoulli, in his *Ars Conjectandi*. This author says, that, the different combinations of the above verse, leaving out the spondees, and admitting those which have no cæsura, amount exactly to 3312. The method by which the enumeration was made may be seen in the above work.

The same question has been proposed respecting the following verse of Thomas Lansius:

Mars, mors, sors, lis, vis, styx, pus, nox, fex,
mala, crux, fraus.

It may be easily found, retaining the word *mala* in the antepenultima place, in order to preserve the measure, that this verse is susceptible of 399168000 different arrangements.

PROBLEM III.

Of the combinations which may be formed with squares divided by a diagonal into two differently coloured triangles.

We are told by Father Sebastian Truchet, of the Royal Academy of Sciences, in a memoir printed among those of the year 1704, that having seen, during the course of a

* See also Vossius de Scient. Math. cap. vii.

tour which he made to the canal of Orleans, some square porcelain tiles, divided by a diagonal into two triangles of different colours, destined for paving a chapel and some apartments, he was induced to try in how many different ways they could be joined side by side, in order to form different figures. In the first place, it may be readily seen that a single square (Plate II.) according to its position can form four different figures, which however may be reduced to two, as there is no other difference between the first and the third, or between the second and the fourth, than what arises from the transposition of the shaded triangle into the place of the white one.

Now, if two of these squares be combined together, the result will be 64 different ways of arrangement; for, in that of two squares, one of them may be made to assume four different situations, in each of which the other may be changed 16 times. The result therefore will be 64 combinations, as may be seen in the plate.

We must however observe, with Father Sebastian, that one half of these combinations are only a repetition of the other, in a contrary direction, which reduces them to 32; and if attention were not paid to situation, they might be reduced to 10.

In like manner, we might combine three, four, five, &c, squares together, and in that case it would be found, that three squares are capable of forming 128 figures; that four could form 256, &c.

The immense variety of compartments which arise, in this manner, from so small a number of elements is really astonishing. Father Sebastian gives thirty different kinds, selected from a hundred; and these even are only a very small part of those which might be formed. Some of the most remarkable of them are exhibited in the 2d plate.

In consequence of Father Sebastian's memoir, Father Douat, one of his associates, was induced to pursue this subject still farther, and to publish, in the year 1722, a large work*, in which it is considered in a different manner. In this work it may be seen that four squares, each divided into two triangles of different colours, repeated and changed in every manner possible, are capable of forming 256 different figures; and that these figures themselves, taken two and two, three and three, and so on, will form a prodigious multitude of compartments, engravings of which occupy the greater part of the book.

It is rather surprising that this idea should have been so little employed in architecture; as it might furnish an inexhaustible source of variety in pavements, and other works of the like kind. However this may be, it forms the object of a pastime, called by the French *Jeu du Parquet*. The instrument employed for this pastime, consists of a small table, having a border round it, and capable of receiving 64 or a hundred small squares, each divided into two triangles of different colours, with which people amuse themselves in endeavouring to form agreeable combinations.

CHAPTER IX.

Application of the Doctrine of Combinations to Games of Chance and to Probabilities.

THOUGH nothing, on the first view, seems more foreign to the province of the mathematics than chance, the powers of analysis have, as we may say, enchained this Proteus, and subjected it to calculation. It has found means to measure the different degrees of probability;

* It is entitled *Méthode pour faire une infinité de dessins différents, avec des carreaux mi-partis de deux couleurs par une ligne diagonale; ou, Observations du P. D. Douat, religieux Carme de la P. de T. sur un Memoire inseré dans l'Hist. de l'Acad. royale des Sciences de Paris, année* 1704, *par le P. S. Truchet, religieux du même ordre, Paris* 1722, in 4to.

and this has given birth to a curious branch of the mathematics, the principles of which we shall here explain.

When an event can take place different ways, it is evident that the probability of its happening in a certain determinate manner, will be greater when, of the whole of the ways in which it can happen, the greater number determine it to happen in that manner. In a lottery, for example, every one knows that the probability or hope of obtaining a prize, is greater according as the number of prizes is greater, and as the total number of the tickets is less. The probability, therefore, of an event, is in the compound ratio of the number of the cases which can produce it, taken directly, and of the total number of those according to which it may be varied, taken inversely; consequently it may be expressed by a fraction, having for its numerator the number of the favourable cases, and for its denominator the whole of the cases.

Thus, in a lottery consisting of a thousand tickets, 25 of which only are prizes, the chance of obtaining one of the latter will be represented by $\frac{25}{1000}$, or $\frac{1}{40}$; if the number of the prizes were 50, this probability would be double, for in that case it would be equal to $\frac{1}{20}$; but, on the other hand, if the whole number of tickets, instead of a thousand, were two thousand, the probability would be only one half of the former, that is $\frac{1}{80}$. If the whole number of tickets were infinitely great, the number of prizes still remaining the same, the probability would be infinitely small; and if the whole number of tickets were prizes, it would become certainty, and in that case would be expressed by unity.

Another principle of this theory, necessary to be here explained, the enunciation of which will be sufficient to show the truth of it, is as follows:

We play an equal game, when the money deposited is in direct proportion to the probability of gaining the stake;

for, to play an equal game, is nothing else than to deposit a sum so proportioned to the probability of winning, that, after a great number of throws or games, the player may find himself nearly at par; but for this purpose, the sums deposited must be proportioned to the degree of probability, which each of the players has in his favour. Let us suppose, for example, that A bets against B on a throw of the dice, and that the chances are two to one in favour of A; the game will be equal if, after a great number of throws, the parties separate nearly without any loss; but as there are two chances in favour of A, and only one in favour of B, after three hundred throws A will have gained nearly two hundred, and B one hundred; A therefore ought to deposit 2 and B only one, for by these means, as A in winning two hundred throws will gain 200, B in winning a hundred throws will gain 200 also. In such cases therefore, it is said that two to one may be betted in favour of A.

PROBLEM I.

In tossing up, what probability is there of throwing a head several times successively, or a tail; or, in playing with several pieces, what probability is there that they will be all heads, or all tails?

In this game, which is well known, it is evident, 1st, That as there is no reason why a head should come up rather than a tail, or a tail rather than a head, the probability that one of the two will be the case is equal to $\frac{1}{2}$, or an equal bet may be taken, for or against.

But if the game were for two throws, and any one should bet, that a head will come up twice, it must be observed, that all the combinations of head or tail, which can take place in two successive throws with the same piece, are *head, head; head, tail; tail, head; tail, tail;* one of which only gives head, head. There is therefore only one case in 4 which can make the person win who bets to throw a head twice in succession; consequently the probability of this event is only $\frac{1}{4}$; and he who bets in favour of two heads, ought to deposit a crown, and the person who bets against him ought to deposit three; for the latter has three chances of winning, while the former has only one. To play an equal game then, the sums deposited by each, ought to be in this proportion.

It will be found also, that he who bets to throw a *head* three times in succession, will have in his favour only one of the eight combinations of head and tail, which may result from three throws of the same piece. The probability of this event therefore, is $\frac{1}{8}$, while that in favour of his adversary will be $\frac{7}{8}$. Consequently, to play an equal game, he ought to stake 1 against 7.

It is needless to go over all the other cases; for it may be easily seen, that the probability of throwing a *head* four times successively, is $\frac{1}{16}$; five times successively, $\frac{1}{32}$, &c.

It is unnecessary also to enumerate all the different combinations which may result from *head* or *tail;* but in regard to probabilities, the following simple rule may be employed.

The probabilities of two or more single events being known, the probability of their taking place all together may be found, by multiplying together the probabilities of these events, considered singly.

Thus the probability of throwing a head, considered singly, being expressed at each throw by $\frac{1}{2}$, that of throwing it twice in succession, will be $\frac{1}{2} \times \frac{1}{2}$ or $\frac{1}{4}$; that of throwing it three times, in three successive throws, will be $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$, or $\frac{1}{8}$, &c.

2d. The problem, to determine the probability of throwing, with two, three, or four pieces, all *heads* or all *tails,* may be resolved by the same means. When two pieces are tossed up, there are four combinations of *head* and *tail*, one of which only is all heads. When three pieces are tossed up together, there are 8, one of which only gives all *heads*, &c. The probabilities of these cases therefore, are the same as those of the cases similar to them, which we have already examined.

It may be easily seen indeed, without the help of analysis, that these two questions are absolutely the same; and the following mode of reasoning may be employed to prove it. To toss up the two pieces A and B together, or to toss them up in succession, giving time to A, the first, to settle before the other is tossed up, is certainly the same thing. Let us suppose then, that when A, the first, has settled, instead of tossing up B, the second, A, the first, is taken from the ground, in order to be tossed up a second time; this will be the same thing as if the piece B had been employed for a second toss; for by the supposition they are both equal and similar, at least in regard to the chance whether head or tail will come up. Consequently, to toss up the two pieces A and B, or to toss up twice in succession the piece A, is the same thing. Therefore, &c.

3d. We shall now propose the following question: What may a person bet, that, in two throws, a *head* will come up at least once? By the above method it will be found, that the chances are 3 to 1. In two throws, indeed, there are four combinations, three of which give at least a head once in the two throws, and one only which gives all tails; hence it follows, that there are three combinations in favour of the person who bets to bring a head once in two throws, and only one against him.

PROBLEM II.

Any number of dice being given; to determine what probability there is of throwing an assigned number of points.

We shall first suppose that the dice are of the ordinary kind, that is to say, having six faces, marked with the numbers 1, 2, 3, 4, 5, 6; and we shall analyse some of the first cases of the problem, in order that we may proceed gradually to those that are more complex.

1st. *It is proposed to throw a determinate point,* 6 *for example, with one die.*

Here it is evident, that, as the die has six faces, one of which only is marked 6, and as any one of them may as readily come up as another, there are 5 chances against the person who proposes to throw a six at one throw, and only one in his favour.

2d. *Let it be proposed to throw the same point* 6 *with two dice.*

To analyse this case, we must first observe that two dice give 36 different combinations; for each of the faces of the die A, for example, may be combined with each of those of the die B, which will produce 36 combinations. But 6 may be thrown, 1st. by 3 and 3; 2d. by 2 with the die A and 4 with the die B, or by 4 with the die A and 2 with the die B, which, as may be readily seen, forms two distinct cases: 3d. by 1 with the die A and 5 with the die B, or 1 with B and 5 with A, which also gives two cases; and these are all that are possible. Hence there are 5 favourable chances in 36; consequently the probability of throwing 6 with two dies, is $\frac{5}{36}$, and that of not throwing it is $\frac{31}{36}$. This therefore ought to be the ratio of the stakes or money deposited by the players.

By analysing the other cases, it will be found that, of throwing two with two dice, there is one chance in 36; of throwing three, there are 2; of throwing four, 3; of throwing five, 4; of throwing six, 5; of throwing seven, 6; of throwing eight, 5; of throwing nine, 4; of throwing ten, 3; of throwing eleven, 2; and of throwing sixes, 1.

If three dice were proposed, with which it is evident the lowest point would be three, and the highest eighteen, it will be found, by means of a similar analysis, that in 216, the whole number of the throws possible with three dice, there is 1 chance of throwing three; 3 of throwing four; 6 of throwing five, &c: as may be seen in the annexed table, the use of which is as follows.

Table of the different ways in which any point can be thrown with one, two, three, or more dice.

	Number of the dice.					
Number of Points.	I	II	III	IV	V	VI
1	1					
2	1	1				
3	1	2	1			
4	1	3	3	1		
5	1	4	6	4	1	
6	1	5	10	10	5	1
7		6	15	20	15	6
8		5	21	35	35	21
9		4	25	56	70	56
10		3	27	80	126	126
11		2	27	104	205	252
12		1	25	125	305	456
13			21	140	420	756
14			15	146	540	1161
15			10	140	651	1666
16			6	125	735	2247
17			3	104	780	2856
18			1	80	780	3431
19				56	735	3906
20				35	651	4221
21				20	540	4332
22				10	420	4221
23				4	305	3906
24				1	205	3431
25					126	2856

If it be required, for example, to find in how many ways 13 can be thrown with three dice, we must look in the first vertical column, on the left, for the number 13, and at the top of the table for 3, the number of the dice, and in the square below, opposite to 13, will be found 21, the number of ways in which 13 may be thrown with three dice. In like manner, it will be found, that with 4 dice, it may be thrown 140 ways; with five dice, 420; &c.

When it is once known, how many ways a point can be thrown with a certain number of dice, the probability of throwing it may be easily found: nothing is necessary but to form a fraction having for its numerator the number of ways in which the point can be thrown, and for denominator the number 6, raised to that power indicated by the number of dice; as the cube of 6, or 216, for three dice; the biquadrate, or 1296, for four dice; &c.

Thus, the probability of throwing 13 with three dice, is $\frac{21}{216}$; of throwing it with four, $\frac{140}{1296}$; &c.

Various other questions may be proposed concerning the throwing of dice, a few of which we shall here examine.

1st. *When two players are engaged; to determine the advantage or disadvantage of the person who undertakes to throw a certain face, that for example marked* 6, *in a certain number of throws.*

Let us suppose that he undertakes it at one throw: to find the probability of his succeeding, it must be considered, that he who holds the die, has only one chance of winning, and five of losing; consequently to undertake it at one throw, he ought to stake no more than 1 to 5. There is therefore a great disadvantage in undertaking, on an even bet, to throw six at a single throw of one die.

To determine the probability of throwing the face marked 6 in two throws with a single die, we must observe, as has been already said, in regard to tossing up, that this is the same thing as to undertake, in throwing two dice together, that one of them shall have the side marked 6 uppermost. He then who holds the dice has only 11 chances, or combinations, by which he can win; for he may throw 6 with the first die and 1, 2, 3, 4 or 5 with the second; or 6 with the second die and 1, 2, 3, 4 or 5 with the first, or 6 with each die. But there are 25 combinations or chances unfavourable to his winning, as may be seen in the following table:

1, 1	2, 1	3, 1	4, 1	5, 1
1, 2	2, 2	3, 2	4, 2	5, 2
1, 3	2, 3	3, 3	4, 3	5, 3
1, 4	2, 4	3, 4	4, 4	5, 4
1, 5	2, 5	3, 5	4, 5	5, 5

Hence it may be concluded, that he who undertakes to throw a six with two dice, ought to stake no more than 11 to 25; and consequently, that it would be disadvantageous to do it on equal terms.

It must here be observed that 36, the number of all the chances or combinations possible in two throws of the dice, is the square of 6, which is the number of the faces of one die; and that 25, the number of the chances unfavourable to the person who undertakes to throw a determinate face, is the square of 5, or of 1 less than the same number 6. The number of the favourable chances therefore, in this case, is equal to the difference of the squares 36 and 25, or of the square of the number of the faces of one die, and of that of the faces of the same die less one.

In the case of undertaking to bring a 6 in three throws with one die, we must consider in like manner, that this is the same thing as to undertake that, in throwing three dice at once, one of them shall bring a 6; but of the 216 combinations, which result from three dice, there are 125 without a 6, and 91 among which there is at least one 6; consequently, he who engages to throw a 6, either in three throws with one die, or one throw with three dice, ought to bet no more than 91 to 125; and it would be disadvantageous to undertake it on equal terms.

It is here to be observed, that the number 91 is the difference of the cube of the number of the faces of one die, viz, 216, and of 125, the cube of the same number less unity, or of 5. Hence it appears that, in general, to find the probability of throwing a determinate face, in a certain number of throws, or at one throw with a certain number of dice, we must raise 6, the number of the faces of one die, to that power which is indicated by the number of throws agreed on, or by the number of dice to be thrown at one time; we must then raise 6 less unity, or 5, to the same power, and subtract it from the former; the remainder with this power of 5 will be the respective number of chances for winning or losing.

Thus, if a person should bet to throw at least one 3 with four dice, we must raise 6 to the 4th power, which is 1296, and subtract from it the 4th power of 5, or 625; the remainder 671 will be the number of chances for winning, and 625 that of the chances for losing; consequently there will be an advantage in laying an even bet.

It is advantageous also to undertake, on an even bet, to throw any determinate point, for example 3, in five throws, or with five dice; for if from the 5th power of 6, which is 7776, we deduct the 5th power of 5, or 3125, the remainder 4651 will be the number of favourable chances, and 3125 that of the unfavourable. Consequently, to play an equal game, he who bets on throwing the above point, ought to deposit 4651 to 3125, or nearly 3 to 2.

3d. *In how many throws may one bet, on equal terms, to throw a determinate doublet, for example, sixes, with two dice?*

It has been already shown, that the probability of not throwing sixes with two dice, is $\frac{35}{36}$; consequently the probability of their not coming up in two throws, will be the square of that fraction; in three throws the cube, &c. But as the powers of every number greater than unity, however small the excess, go on always increasing, those of a number less than unity, however small the defect, go on always decreasing; the consecutive powers therefore of $\frac{35}{36}$ will go on always decreasing. Now let us conceive $\frac{35}{36}$ to be raised to such a power as to be equal to $\frac{1}{2}$; it will be found that the 24th power of $\frac{35}{36}$ is somewhat greater than $\frac{1}{2}$; and that the 25th power is somewhat less *; hence it follows, that one may lay an even bet with some advantage, that another will not bring sixes in 24 throws with two dice, but that there is some disadvantage in taking an even bet that they will not come up in 25 throws. Consequently he who bets on throwing sixes in 24 throws, does so with disadvantage; but if he lays an even bet that they will come up in 25 throws, the advantage is in his favour.

4th. *What probability is there of throwing any determinate doublet, for example, two threes, in one throw with two or more dice?*

To determine this question, we must first observe, that he who undertakes to throw two threes with two dice, has only one favourable chance, in the 36 chances or combinations given by two dice; and it thence follows that he ought to bet no more than 1 to 35.

In the case of three dice, it will be found that he ought to bet no more than 16 to 200; for the number of chances or combinations possible with three dice is 216. But when it is proposed to throw two threes with three dice, they may come up 16 different ways; for in the 36 com-

* Let n be the exponent of that power of $\frac{35}{36}$ which is equal to $\frac{1}{2}$; that is to say, let $\frac{35^n}{36^n}$ be equal to $\frac{1}{2}$. As the unknown quantity n is in the exponent, it must be disengaged from it, which may be done by means of logarithms. For if $\frac{35^n}{36^n}=\frac{1}{2}$, by taking the logarithms we shall have n log. 35 — n log. 36 = log. $\frac{1}{2}$, or = — log. 2; for log. $\frac{1}{2}$ = — log. 2. Hence n log. 35 — n log. 36 = — log. 2, or log. 2 = n log. 36 — n log. 35. Therefore, $n=\frac{\text{log. } 2}{\text{log. } 36-\text{log. } 35}$. Which gives $n=24{\cdot}605$, or $24\frac{6}{10}$ nearly.

binations of the two dice A and B, all those in which one 3 only is found, as 1, 3; 3, 1; &c, being 10 in number, by combining with the side marked 3 of the die C, give two threes. Besides, the combination 3, 3 of the dice A and B, by combining with one of the six faces of the third C, will give two threes. Here then we have 16 ways of throwing two threes with three dice, which give 16 favourable chances in 216. Consequently, the probability of throwing two threes with three dice, is $\frac{16}{216}$; and no more ought to be betted on the success of that event than 16 to 200, or 2 to 25.

If the probability of throwing two threes with four dice be required; we shall find that it is expressed by $\frac{171}{1296}$. For, of the 1296 combinations of the faces of four dice, there are 150 which give 2 threes; 20 that give 3, and one that gives 4, making altogether 171 throws, which give 2 or 3 or 4 threes. Consequently, no more than 171 to 1125, or about 1 to $6\frac{3}{5}$, ought to be betted on throwing, at least, once threes with four dice.

In the last place, if the probability of throwing any doublet, at one throw, with two or more dice, be required; it may be easily determined by the preceding method of calculation: for if an indeterminate doublet be proposed, it is evident that the probability is six times as great as when an assigned doublet is proposed; and therefore we have only to multiply the probabilities already found by 6. The probability therefore with two dice, will be $\frac{6}{36}$ or $\frac{1}{6}$; with three dice, $\frac{96}{216}$ or $\frac{4}{9}$; with four dice, $\frac{1026}{1296}$ or $\frac{19}{24}$; &c. So that there is an advantage in taking an even bet to throw at least one doublet with four dice.

PROBLEM III.

Two persons sit down to play for a certain sum of money; and agree that he who first gets three games shall be the winner. One of them has got two games, and the other one; but being unwilling to continue their play, they re-

solve to divide the stake: how much of it ought each person to receive?

This problem is one of the first that engaged the attention of Pascal, when he began to study the calculation of probabilities. It was resolved by Fermat, a celebrated geometrician, to whom he proposed it, by a different method, viz that of combinations: we shall here give both.

It is evident that each of the players, in depositing his money, resigns all right to it; but, in return, each has a right to what chance may give him; consequently when they give over playing, the stake ought to be divided in proportion to the probability each had of winning the whole sum, had they continued.

Case 1st. This proposition may be determined by the following reasoning. Since the first player wants one game to be out, and the second two, it may be readily perceived, that if they continue their play, and if the second should win one game, he would want, in the same manner as the first, one game to be out; and in that case, the two players being equally advanced, their hopes or chances of winning, would be equal. This being supposed, they would have an equal right to the stake, and consequently each ought to have an equal share of it.

It is evident therefore, that if the first should win the game about to be played, the whole money deposited would belong to him; and that if he lost it, he would have a right only to the half. But the one case being as probable as the other, the first has a right to the half of both these sums taken together. But together they make $\frac{3}{2}$; the half of which is $\frac{3}{4}$; and this is the share of the stake belonging to the first player; consequently that belonging to the second is only $\frac{1}{4}$.

Case 2d. The solution of the first case will enable us to solve the second, in which we suppose that the first player wants one game to be out, and the second three; for if the first should win one game, he would be entitled to the whole stake, and if he lost one game, by which means the second would want only two games to be out, $\frac{3}{4}$ of the money would belong to the former, as the parties would then be in the situation alluded to in the preceding case. But as both these events are equally probable, the first ought to have the half of the two sums taken together, or the half of $\frac{7}{4}$, that is $\frac{7}{8}$: the remainder $\frac{1}{8}$ will therefore be what belongs to the second.

Case 3d. It will be found, by the like reasoning, if we suppose two games wanting to the first player, and three to the second, that on ceasing to play, they ought to divide the stake in such a manner, that the former may have $\frac{11}{16}$, and the latter $\frac{5}{16}$.

Case 4th. If four games were to be played; and if the first wanted only two games, and the second four, the money ought to be divided in such a manner, that the former should have $\frac{13}{16}$, and the latter $\frac{3}{16}$.

But we may dispense with the above reasoning, and employ the following general rule, deduced from it, which is to be applied by means of the arithmetical triangle. Enter that diagonal of the arithmetical triangle the order of which is equal to the number of the games wanting to both players. As this number in the first case is 3, we must enter the third diagonal of the triangle; then because the first player wants only one game, we must take the first number of that diagonal; but because two are wanting to the second, we must take the sum of the two first numbers, which will give 3. These two numbers therefore, 1 and 3, will indicate, that the stake ought to be divided in the same proportion: consequently the first player ought to have $\frac{3}{4}$, and the second $\frac{1}{4}$.

As this rule may be easily applied to every other case whatever, we shall enlarge no further on the subject.

The second method of resolving problems of this kind, which is that of combinations, is as follows:

To resolve, for example, the fourth case, where, according to the supposition, the first player wants two games to be out, and the second four, so that together they want six games; take unity from that sum, and because 5 remain, we shall suppose the five similar letters, *a a a a a*, favourable to the first player, and the five following, *b b b b b*, favourable to the second. These letters must be combined, as in the following table, where, of the 32 combinations which they form, the first 26, towards the left, where *a* occurs at least twice, will indicate the number of chances which the first has of winning; and the last 6, towards the right, in which *a* never occurs oftener than once, will indicate those favourable to the second.

a a a a a	*a a a b b*	*a a b b b*	*a b b b b*
a a a a b	*a a b b a*	*a b b b a*	*b b b b a*
a a a b a	*a b b a a*	*b b b a a*	*b a b b b*
a a b a a	*b b a a a*	*a b a b b*	*b b a b b*
a b a a a	*a a b a b*	*a b b a b*	*b b b a b*
b a a a a	*a b a a b*	*b b a a b*	*b b b b b*
	b a a a b	*b a a b b*	
	b a a b a	*b a b b a*	
	b a b a a	*b b a b a*	
	a b a b a	*b a b a b*	

The expectation therefore of the first player, will be to that of the second, as 26 to 6, or as 13 to 3.

In like manner, to resolve the case where the first player is supposed to have won three games, and the other none, as he must win who first gets four games, the number of the games wanting to both will be 5, which being diminished by unity, will give 4. We must then examine in how many different ways the letters *a* and *b* can be combined four and four, which will be found to be 16, viz:

a a a a	*a a b b*	*a b b b*	
a a a b	*a b a b*	*b a b b*	*b b b b*
a a b a	*b a a b*	*b b a b*	
a b a a	*a b b a*	*b b b a*	
b a a a	*b a b a*		
	b b a a		

But, of these 16 combinations, it is evident there are 15 where a is found at least once; which indicates that there are 15 combinations or chances favourable to the first player, and one favourable to the second. Consequently they ought to divide the stake in the ratio of 15 to 1, or the former ought to have $\frac{15}{16}$ of it, and the latter $\frac{1}{16}$.

PROBLEM IV.

Of the Genoese Lottery.

All persons are acquainted with the nature of lotteries, a kind of institution which originated in Italy, and which was afterwards introduced into other countries of Europe. It took its rise at Genoa, where it had long been customary to choose annually by ballot five members of the senate, which was composed of 90 persons, in order to form a particular council. Some idle persons took this opportunity of laying bets, that the lot would fall on such or such senators. The government then seeing with what eagerness people interested themselves in these bets, conceived the idea of establishing a lottery on the same principle; which was attended with so great success, that all the cities of Italy wished to participate in it, and sent large sums of money to Genoa for that purpose. The same motive, and that no doubt of increasing the revenues of the church, induced the pope to establish one of the same kind at Rome, the inhabitants of which became so fond of this species of gambling, that they often deprived themselves and their families of the necessaries of life, that they might have money to lay out in the lottery. Many of them also indulged in every kind of foolery that credulity or superstition could inspire, in order to obtain fortunate numbers.

The analysis of this kind of lottery is reduced to the solution of the following problem.

Ninety numbers being given, five of which are to be drawn by chance; it is required to determine what probability there is that, among these five, there will be one, two, three, four, *or five numbers, which any one has chosen from among the* 90?

It may be readily seen, that if one determinate number only were proposed, and that if no more than one number were to be drawn from the wheel, the adventurer would have only one favourable chance in the 90; but as five numbers are drawn from the wheel, this quintuples the chance favourable to the adventurer, so that he has five favourable chances in the ninety. His probability therefore of winning, is $\frac{1}{18}$; and, to play an equal game, the stakes ought to be in the same ratio; or, what amounts to the same thing, the proprietor of the lottery ought to reimburse the price of the ticket 18 times.

To determine what probability there is, that two numbers selected will both come up, we must first find how many combinations may be produced by 90 numbers, taken two and two. In treating on combinations we have already shown, that in this case they amount to 4005; but as five numbers are drawn from the wheel, and as these five numbers, combined together two and two, give 10 twos, it thence results that, in these 4005 chances, there are only 10 favourable to the adventurer. The probability therefore, that the two numbers selected may be among those drawn from the wheel, will be expressed by $\frac{10}{4005}$ or $\frac{1}{400\frac{1}{2}}$. For this reason the proprietor of the lottery ought to give the adventurer, in case he should win, $400\frac{1}{2}$ times the price of the ticket.

To determine what probability there is, that three numbers selected will come up among the five drawn from the wheel, we must find how many ways 90 numbers can be combined three and three, or how many threes they make. These combinations amount to 117480; but as the five numbers drawn from the wheel form 10 threes, the adventurer has 10 favourable chances in 117480, and the probability in his favour is $\frac{10}{117480}$ or $\frac{1}{11748}$. To risk his money therefore on equal terms, the prize ought to be 11748 times the price of the ticket.

In the last place, it will be found that in 511038 chances, there is only one favourable to the person who should bet that four determinate numbers will come up; and 1 in 43949268 favourable to the person who should bet that five determinate numbers will be the five drawn; consequently, in the last case, to risk his money on equal terms, according to mathematical strictness, the adventurer, should he be successful, ought to receive nearly 44 millions of times the money which he lays out.

PROBLEM V.

A certain person, whom we shall call A, *playing at the game of thirteen, bets, that in turning up the cards successively, according to their order, ace, deuce, tré, &c, to the king, which is the last, he will turn up one card, at least, which he has named: what probability has* A *in his favour?*

To enable the reader to comprehend this problem, it will be necessary to explain the nature of the game to which the author alludes. The players having cut for the cards, we shall suppose that A has them, and that the players are any number at pleasure. A then takes the pack, consisting of 52 cards, and, when they have been shuffled, turns them up one after the other, calling one when he turns up the first, two when he turns up the second, and so on to the thirteenth, which is represented by a king. If, in all this series of cards, he turns up none in the order in which he named them, he pays to each of the players what they deposited, and resigns the cards to the next person on his right.

But if it should happen that he turns up, in the series of 13 cards, any one card which he has named, that is to say, if he turns up an ace at the time when he calls out one, or a two when he calls out two, and so on, he gets the whole stake, and begins as before, calling out one, then two, &c.

It may sometimes happen that A, after winning several times, and beginning again at one, has not a sufficient number of cards in his hand to go on to 13; in that case, when the cards are out, they must be shuffled and cut, and he must then take from the pack the number of cards sufficient for him to continue the game, beginning where he left off: that is to say, if in turning up the last card he named seven, in turning up the first of the new cards he must call out eight, then nine, and so on to 13, unless he wins before; in which case he must begin once more, calling out one, then two, and so on as already explained.

As it would be too tedious to enter into a complete analysis of this game, we shall only observe, that according to Montmort, if A holds only two cards, the probability of his winning is $\frac{1}{2}$; if he has three, it is $\frac{2}{3}$; if four, it is $\frac{5}{8}$; and in the last place, if he has 13, it is $\frac{1093339663}{1729728000}$; so that, to play an equal game, A ought to bet somewhat less than 11 to 6.

PROBLEM VI.

A and B playing at piquet; A is first in hand, and has no ace: what probability is there that he will get one, or two, or three, or four?

It is well known that at this game 12 cards are dealt to each of the players, and that 8 remain in the pack, of which the first takes 5, and the last 3. This being premised, it will be found that A's chance to have any one

ace is	$\frac{455}{969}$
to have two	$\frac{270}{969}$
to have three	$\frac{30}{969}$
to have four	$\frac{1}{969}$

the sum of all these is $\frac{756}{969}$, which is equal to $\frac{252}{323}$.

Hence it follows, that the probability of his having an ace among the five cards he has to take in, is $\frac{252}{323}$, the difference between which numbers is 71, so that one may bet 252 to 71 that A will take in some of the aces. But let us suppose that A is last in hand; in that case it is required how much he may bet that he will have at least one ace among his three cards?

The probability of A having an ace among his three cards is . . . . $\frac{8}{19}$ or $\frac{120}{285}$
of having two it is . . . . $\frac{24}{285}$
of having three . . . . $\frac{1}{285}$
the sum of all which is $\frac{145}{285}$ or $\frac{29}{57}$.

Consequently, the probability that he will have either one, or two, or three indeterminately, is $\frac{29}{57}$. A may therefore take an equal bet with advantage, that he will have one of the aces, for the ratio of the stakes would be 29 to 28.

PROBLEM VII.

At the game of whist, what probability is there, that the four honours will not be in the hands of any two partners?

De Moivre, in his Doctrine of Chances, shows that the chance is nearly 27 to 2 that the partners, one of whom deals, will not have the four honours.

That it is about 23 to 1 that the other two partners will not have them.

That it is nearly 8 to 1 that they will not be found on any one side.

That one may bet about 13 to 7, without disadvantage, that the partners who are first in hand will not count honours.

That about 20 to 7 may be betted, that the other two will not count them.

And, in the last place, that it is 25 to 16, that one of the two sides will count honours, or that they will not be equally divided.

PROBLEM VIII.

Of the game of the American Savages.

We are told by Baron de la Hontan, in his *Voyages en Canada*, that the Indians play at the following game: they have 8 nuts, black on the one side and white on the other; these they throw into the air, and if it happens, when they fall to the ground, that the black are odd, the player wins the stake; if they are all black, or all white, he wins the double; but if there are an equal number of each, he loses.

M. de Montmort, who analysed this game, finds, that he who tosses up the nuts, has an advantage, which may be estimated at $\frac{3}{256}$; and that to render the game equal, he ought to deposit 22 when his adversary stakes 21.

PROBLEM IX.

Of the game of Backgammon.

The game of backgammon is one of those where the spirit of combination is displayed in a very striking manner, and where it is of great utility to know, at every throw, what may be hoped or feared from the succeeding throws, whether your own, or those of your adversary. The chances in this game, like those in others, may be appreciated mathematically; but we shall here confine ourselves to a small number of examples, selected from those easiest to be comprehended.

I. *A, being at play at backgammon, is obliged to make a blot; now his throw is such, that he can make it either where his adversary B may take it up with a single ace, or where he can take it up by throwing seven in any manner: the question is, where should he make the blot?*

As the number of chances for throwing one ace or more, is 11, and the number of chances for throwing seven in any manner, are but 6, it will be safest to make the blot where it may be taken up by throwing 7.

II. *Whether it is safer to make a blot, at backgammon, where it may be taken up by an ace, or where it may be taken up by a tré?*

The number of chances for throwing one ace or more, and those for throwing one tré or more, are each 11; but there are 2 chances for throwing deux ace, or 3; it will therefore be safer to make the blot where it can be taken up only by an ace.

The following table will shew the chances of taking up a single blot however situated.

No. of points to hit	Chances	Total chances	No. of points to hit	Chances	Total chances
1	11	11	7	6	6
2	11 + 1	12	8	5	5
3	11 + 2	13	9	4	4
4	11 + 3	14	10	3	3
5	11 + 4	15	11	2	2
6	11 + 5	16	12	1	1

Hence, if a blot is liable to be hit by any one face of the die, the mean probability of hitting it will be $\frac{11 + 16}{2 \times 36}$ $= \frac{27}{72} = \frac{3}{8}$ nearly.

III. *If two blots be made at backgammon, so as to be hit by two different faces of the die, what is the probability of hitting one or both of them?*

By the first table it will appear, that the probability of throwing one or more, of any two given faces, is $\frac{20}{36}$. But, besides this, one or both the blots may be at length hit by the two dice, and the probability in this case will be different, according to the number of points that will hit them, as in the following table:

Faces to hit	Chances	Total chances	Faces to hit	Chances	Total chances
1.2	20 + 1	21	2.6	20+1+5	26
1.3	20 + 2	22	3.4	20+2+3	25
1.4	20 + 3	23	3.5	20+2+4	26
1.5	20 + 4	24	3.6	20+2+5	27
1.6	20 + 5	25	4.5	20+3+4	27
2.3	20+1+2	23	4.6	20+3+5	28
2.4	20+1+3	24	5.6	20+4+5	29
2.5	20+1+4	25			

Hence the probability of hitting two such blots, will be at a medium $\frac{21 + 29}{2 \times 36} = \frac{25}{36}$.

IV. *If there be three blots, so situated as to be hit by three different faces; the probability of hitting one or more of them is required.*

The first table will give the probability of hitting one or more of the blots with a single face or faces. But, besides this, there will be the probability of hitting one or more of the blots with two dice, the least of which will be when the given faces are 1, 2, 3, which have $1 + 2 = 3$ such chances, and the greatest when the given faces are 4, 5, 6, which have $3 + 4 + 5 = 12$ such chances; the medium of these, viz. $\frac{3+12}{2} = \frac{15}{2}$, being added to 27, will make the whole probability about $27 + \frac{15}{2} = \frac{69}{2}$ which divided by the common denominator 36, becomes $\frac{69}{72} = \frac{23}{24}$.

Hence, if a player at backgammon makes 3 blots, which are severally within the reach of being hit by a single face of the die, it is almost a certainty that one of them at least will be hit.

PROBLEM X.

A mountebank at a country fair amused the populace with the following game: he had 6 dice, each of which was marked only on one face, the first with 1, the second with 2, and so on to the 6th, which was marked 6; the person who played gave him a certain sum of money, and he engaged to return it a hundred-fold, if in throwing these six dice, the six marked faces should come up only once in 20 throws. If the adventurer lost, the mountebank offered a new chance on the following conditions: to deposit a sum equal to the former, and to receive both the stakes in case he should bring all the blank faces in 3 successive throws.

Those unacquainted with the method to be pursued in order to resolve such problems, are liable to reason in an erroneous manner on dice of this kind; for, observing that there are five times as many blank as marked faces, they thence conclude that it is 5 to 1 that the person who throws them will not bring any point. They are however mistaken, as the probability, on the contrary, is near 2 to 1 that they will not come up all blanks.

If we take only one die, it is evident that it is 5 to 1 that the person who holds it will throw a blank; but if we add a second die, it may be readily seen, that the marked face of the first may combine with each of the blank faces of the second, and the marked face of the second with each of the blank faces of the first; and, in the last place, the marked face of the one with the marked face of the other: consequently, of the 36 combinations of the faces of these two dice, there are 11 in which there is at least one marked face. But, as we have already observed, this number 11 is the difference of the square of 6, the number of the faces of one die, and of the square of the same number diminished by unity, that is to say of 5.

If a third die be added, we shall find, by the like analysis, that of the 216 combinations of three dice, there are 91 in which there is at least one marked face; and 91 is the difference of the cube of 6 or 216, and the cube of 5 or 125; the result will be the same in regard to the more complex cases; and hence we may conclude that, of the 46656 combinations of the faces of the 6 dice in question, there will be 31031 in which there is at least one marked face, and 15625 in which all the faces are blank; consequently, the chance is 2 to 1 that some point at least will be thrown; whereas, by the above reasoning, it would appear that 5 to 1 might be betted on the contrary being the case.

This example may serve to shew how diffident we ought to be in regard to the ideas which occur on the first consideration of subjects of this kind; and it may be added that, in this case, our reasoning is confirmed by experience. But to return to the problem: it is evident that, of the 46656 combinations of the faces of 6 dice, there is only one which gives the 6 marked faces uppermost; the probability therefore of throwing them at one throw, is expressed by $\frac{1}{46656}$, and as the adventurer was allowed 20 throws, the probability of his succeeding was only $\frac{20}{46656}$, which is nearly equal to $\frac{1}{2332}$. To play an equal game therefore, the mountebank should have engaged to return 2332 times the money. But he offered only 100 times the stake, that is, about the 23d part of what he ought to have offered, to give an equal chance, and consequently he had an advantage of 22 to 1.

The chance offered to those who might lose was a mere deception; for the proposer artfully availed himself of that propensity which every man who had not sufficiently examined the subject, would have to adopt the false reasoning above mentioned; and the adventurer would have the less hesitation to accept the offer as it would seem that he might bet 5 to 1 on bringing blanks every throw; whereas it is 2 to 1 that the contrary will happen. But the chance of not bringing blanks in one throw, being to that of bringing them, as 2 to 1; it thence follows, that the probability of not bringing them three times successively, is to that of bringing them, as 8 to 1. To play an equal game therefore, the mountebank ought to have staked 7 to 1; consequently, in the chance which he gave to the loser, in a game where he had an advantage of 22 to 1, he had still an advantage of 7 to 1.

PROBLEM XI.

In how many throws with six dice, marked on all their faces, may a person engage, for an even bet, to throw 1, 2, 3, 4, 5, 6?

We have just seen that there are 46656 chances to 1, that a person will not throw these six points with dice marked only on one of their faces; but the case is very different with 6 dice marked on all their faces; and to prove it, we need only observe that the point 1, for example, may be thrown by each of the dice, as well as the 2, 3, &c; which renders the probability of these six points, 1, 2, 3, &c, coming up, much greater.

But to analyse the problem more accurately, we shall observe, that there are 2 ways of throwing 1, 2 with two dice; viz 1 with the die A, and 2 with the die B; or 1 with the die B, and 2 with A. If it were proposed to throw 1, 2, 3 with 3 dice; of the whole of the combinations of the faces of 3 dice, there are 6 which give the points 1, 2, 3; for 1 may be thrown with the die A, 2 with B, and 3 with C; or 1 with A, 2 with C, and 3 with B; or 1 with B, 2 with A, and 3 with C; or 1 with B, 2 with C, and 3 with A; or 1 with C, 2 with A, and 3 with B; or 1 with C, 2 with B, and 3 with A.

It hence appears that, to find the number of ways in which 1, 2, 3 can be thrown with 3 dice, 1, 2, 3 must be multiplied together. In like manner, to find the number of ways in which 1, 2, 3, 4 can be thrown with 4 dice, we must multiply together 1, 2, 3, 4, which will give 24; and, in the last place, to find in how many ways 1, 2, 3, 4, 5, 6 can be thrown with 6 dice, we must multiply together these six numbers, the product of which will be 720.

If the number 46656, which is the combinations of the faces of 6 dice, be divided by 720, we shall have $64\frac{4}{5}$ for the chances to 1, that these points will not come up at one throw; consequently a person may undertake, for an even bet, to bring them in 56 throws; and one may bet more than 2 to 1 that they will come up in 130 throws. In the last place, as the dice may be thrown 130 times and more, in a quarter of an hour, a person may with advantage bet more than 2 to 1, that they will come up in the course of that time.

He therefore who engages, for an even bet, to throw these points in a quarter of an hour, undertakes what is highly advantageous to himself, and equally disadvantageous to his adversary.

PROBLEM XII.

A certain person proposed to play with 7 dice, marked on all their faces, on the following conditions: he who held the dice was to gain as many crowns as he brought sixes; but if he brought none, he was to pay to his adversary as many crowns as there were dice, that is 7. What was the ratio of their chances?

To resolve this problem, we must analyse it in order. Let us suppose then, that there is only one die; in this case it is evident, that as there is only 1 chance in favour of him who holds the die, and 5 against him, the ratio of the stakes ought to be that of 1 to 5. If the first therefore gave a crown every time he did not throw 6, and received only the same sum when a 6 came up, he would play a very unequal game.

Let us now suppose 2 dice. In the 36 combinations, of which the faces of 2 dice are susceptible, there are 25 which give no 6; 10 which give 1, and 1 which gives 2. He therefore who holds the dice has only 11 chances in his favour, 10 of which may each make him gain a crown, and the remaining 1 may make him gain two. His chance then of winning, according to the general rule, will be $\frac{10}{36} + \frac{2}{36}$; and because, if the 25 chances which do not give a 6 should take place, he would be obliged to pay 2 crowns, the chance of his adversary will be $\frac{50}{36}$. Consequently the chance of winning will be to that of losing as $\frac{12}{36}$ to $\frac{50}{36}$, or 12 to 50, or less than 1 to 4.

To determine, in the more complex cases, the chances which give no 6, those which give one, those which give two, &c, it must be observed, that they are always expressed by the different terms of the power of 5 + 1, the exponent of which is equal to the number of the dice. Thus, when there is only one die, the number 5 + 1 expresses, by its first term, that there are five chances without a 6, and one which gives a 6; if there be two dice, as the product of 5 + 1 by 5 + 1, or the square of 5 + 1, is 25 + 10 + 1, the first term 25 indicates that there are 25 chances, in the 36, which give no 6; 10 which give one, and 1 which gives two.

In like manner, as the cube of 5 + 1 is 125 + 75 + 15 + 1, it denotes that, in the 216 combinations of the faces of 6 dice, there are 125 in which there is no 6; 75 in which there is one; 15 in which there are two, and 1 where there are three.

The fourth power of 5 + 1 being 625 + 500 + 150 + 20 + 1, it indicates, in the same manner, that in the 1296 combinations of the faces of four dice, there are 625 without a 6; 500 which give 6; 150 which give two, 20 which give three, and only 1 that gives four.

We shall pass over the intermediate cases, and proceed to that where 7 dice are employed. In this case then it will be found, that the 7th power of 5 + 1 is 78125 + 109375 + 65625 + 21875 + 4375 + 525 + 35 + 1 = 279936. In the 279936 combinations of the faces of 7 dice, there are 78125 which give no 6; 109375 where there is one; 65625 where there are two; 21875 where there are three, &c. But as he who holds the dice would have to pay 7 crowns for each of the first 78125 chances, should they take place, we must consequently, according to the general rule, multiply that number by 7, and divide the product by the sum of all the chances, in order to obtain the chance against him, which is $= \frac{546875}{279936}$. To find the favourable chance, we must multiply each of the other terms by the number of the sixes it presents; add together the different products, and divide the sum by the whole of the chances, or 279936; in this manner we shall have, for the chance in favour of the person who holds the dice, $\frac{325592}{279936}$. His chance of winning, therefore, is to that of losing, as 325592 to 54687; that is to say, he plays a disadvantageous game, or it is 54 to 32, or 27 to 16, or more than 3 to 2, that he will lose.

By a like process it may be found, in the case of eight dice, that the chance of the person who holds them, is to that of his adversary, as 2259488 to 3125000, which is nearly as 3 to 4.

If there were nine dice, the chance of the person who holds them, would be to that of his adversary, nearly as 151 to 175, or nearly 25 to 29.

If there were ten dice, the chance of the former to that of the latter, would be as 101176960 to 97656250, that is to say, nearly as 101 to $97\frac{6}{10}$. The advantage then begins to be in favour of the former, only when the number of the dice is ten; and, to play an equal game, a less number ought not to be employed.

CHAPTER X.

Arithmetical Amusements in Divination and Combinations.

PROBLEM I.

To tell the number thought of by a person.

I.

DESIRE the person, who has thought of a number, to triple it, and to take the exact half of that triple if it be even, or the greater half if it be odd. Then desire him to triple that half, and ask him how many times it contains 9; for the number thought of will contain the double of that number of nines, and one more if it be odd.

Thus if 5 has been the number thought of; its triple will be 15, which cannot be divided by 2 without a remainder. The greater half of 15 is 8; and if this half be multiplied by 3, we shall have 24, which contains 9 twice; the number thought of will therefore be 4 + 1, that is to say 5.

II.

Bid the person multiply the number thought of by itself; then desire him to add unity to the number thought of, and to multiply that sum also by itself; in the last place ask him to tell the difference of these two products, which will certainly be an odd number, and the least half of it will be the number required.

Let the number thought of, for example, be 10, which multiplied by itself gives 100; in the next place 10 increased by 1 is 11, which multiplied by itself makes 121, and the difference of these two squares is 21, the least half of which, being 10, is the number thought of.

This operation might be varied in the second step, by desiring the person to multiply the number by itself, after it has been diminished by unity, and then to tell the difference of the two squares; the greater half of which will be the number thought of.

Thus, in the preceding example, the square of the number thought of is 100, and that of the same number less unity is 81: the difference of these is 19, the greater half of which, or 10, is the number thought of.

III.

Desire the person to add to the number thought of its exact half, if it be even, or its greater half if odd, in order to obtain a first sum; then bid him add to this sum its exact half, or its greater half, according as it is even or odd, to have a second sum; from which the person must subtract the double of the number thought of. Then desire him to take the half of the remainder, or its less half if it be an odd number, and to continue halving the half till he comes to unity. When this is done, count how many sub-divisions have been made, and for the first division retain 2, for the second 4, for the third 8, and so of the rest, in double proportion. It is here necessary to observe, that 1 must be added for each time that the least half was taken, because, by taking the least half, 1 always remains; and that 1 only must be retained when no subdivision could be made; for thus you will have the number, the halves of the halves of which have been taken: the quadruple of that number then will be the number thought of, in case it was not necessary at the beginning to take the greater half, which will happen only when the number thought of is evenly even, or divisible by four; but if the greater half has been taken at the first division, 3 must be subtracted from the above quadruple, or only 2 if the greater half has been taken at the second division, or 5 if it has been taken at each of the two divisions, and the remainder then will be the number thought of.

Thus, if the number thought of has been 4; by adding to it its half, we shall have 6; and if to this we add its half 3, we shall have 9: if 8, the double of the number thought of, be subtracted, there will remain 1, which cannot be halved, because we have arrived at unity; for this reason we must retain 1, the quadruple of which, 4, is the number thought of.

If 5 has been thought of; by adding to it its greater half 3, we shall have 8; and if 4, its half, be added, the sum will be 12; from which if we subtract 10, the double of 5, the number thought of, the remainder will be 2, the half of which is 1; and as we can no longer take the half, because we have arrived at unity, we must retain 2 as there has been one sub-division. If from 8, the quadruple of 2, the number retained, we subtract 3, because in the first division the greater half was taken, the remainder 5 will be the number thought of.

IV.

Desire the person to take 1 from the number thought of, and to double the remainder; then bid him take 1 from this double, and add to it the number thought of. Having asked the number arising from this addition, add 3 to it, and the third of the sum will be the number required.

Let the number thought of be 5; if 1 be taken from it, there will remain 4, the double of which 8 being diminished by 1, and the remainder 7 being increased by 5, the number thought of, the result will be 12: if to this we add 3, we shall have 15, the third part of which, 5, will be the number required.

REMARK.

This method may be varied a great many ways; for instead of doubling the number thought of, after unity has been deducted from it, the person may be desired to triple it; then after he has been desired to subtract unity from that triple, and to add the number thought of, he must add 4 to it, and the $\frac{1}{4}$ of the sum arising from these operations will be the number required.

Let the number required be x; if unity be subtracted from it, the remainder will be $x-1$; multiply this remainder by any number whatever, n, and the product will be $nx-n$; again subtract unity, and we shall have for remainder $nx-n-1$; if x, the number thought of, be then added, the sum will be $(n+1)x-n-1$; and if to this sum we add the above multiplier increased by unity, that is to say, 3 if the first remainder was doubled, 4 if it was tripled, &c, the result will be $(n+1)x$; which being divided by the same number, the quotient will be x, the number required.

Unity, instead of being subtracted from the number thought of, might be added to it; and then, instead of adding, at the end of the operation, the multiplier increased by unity, it ought to be subtracted, after which the remainder may be divided as above.

Let the number thought of, for example, be 7; if unity be added, the sum will be 8, and this sum tripled will give 24; if 1 be still added, we shall have 25, and this sum increased by 7 will make 32; from which if 4 be deducted, because the number thought of was tripled after unity had been added, we shall have 28: one fourth of which will be the number required.

V.

Desire the person to add 1 to the triple of the number thought of, and to multiply the sum by 3; then bid him add to this product the number thought of, and the result will be a sum, from which if 3 be subtracted, the remainder will be decuple of the number required. If 3 therefore be taken from the last sum, and if the cipher on the right be cut off from the remainder, the other figure will indicate the number sought.

Let the number thought of be 6; the triple of which is 18, and if unity be added, it makes 19; the triple of this last number is 57, and if 6 be added it makes 63, from which if 3 be subtracted the remainder will be 60: now if the cipher on the right be cut off, the remaining figure 6 will be the number required.

REMARK.

If 1 were subtracted from the number thought of, the remainder doubled, and the number thought of again added, it would be necessary, after the person had told the result, which would always terminate with 7, to add 3 instead of subtracting it, as in the above operation; and the sum would then be the decuple of the number thought of.

To the foregoing methods given by Montucla, of telling the number that a person has thought of, may be added the following one, by means of the annexed columns of numbers; a method so truly ingenious and scientific, that it is really astonishing in its nature and effects. These six columns of numbers may either be all on one paper as here, but better for deception on as many separate slips. We shall describe the method of constructing these columns of numbers after explaining their uses, as follows.

Bid a person think in his mind of any number he pleases, not higher than 63, the greatest number in the columns. Desire him then to point out to you all the columns in which his number is contained; which he can presently do, on your showing him the columns, one after another, and then withdrawing them away from his sight again. Then, recollecting that the top numbers in the six columns are the geometrical series 1, 2, 4, 8, 16, 32, add together in your mind the figures of this series which are at the tops of all those columns in which the required number is contained; and whatever that sum

I	II	III	IV	V	VI
1	2	4	8	16	32
3	3	5	9	17	33
5	6	6	10	18	34
7	7	7	11	19	35
9	10	12	12	20	36
11	11	13	13	21	37
13	14	14	14	22	38
15	15	15	15	23	39
17	18	20	24	24	40
19	19	21	25	25	41
21	22	22	26	26	42
23	23	23	27	27	43
25	26	28	28	28	44
27	27	29	29	29	45
29	30	30	30	30	46
31	31	31	31	31	47
33	34	36	40	48	48
35	35	37	41	49	49
37	38	38	42	50	50
39	39	39	43	51	51
41	42	44	44	52	52
43	43	45	45	53	53
45	46	46	46	54	54
47	47	47	47	55	55
49	50	52	56	56	56
51	51	53	57	57	57
53	54	54	58	58	58
55	55	55	59	59	59
57	58	60	60	60	60
59	59	61	61	61	61
61	62	62	62	62	62
63	63	63	63	63	63

amounts to, you may infallibly pronounce as the number thought of.

Thus, for example, if the person says his number is contained only in the 2nd, 3rd, and 5th columns; then recollecting that the top numbers of these three columns are 2, 4, 16, the sum of which is 22, you announce, to his astonishment, that the number he thought of is 22. Again, supposing a person thinking of a number, and, looking at the columns, says it is contained only in the 1st, 4th, and 6th; then recollecting that the top numbers of these three columns are 1, 8, 32, the sum of which is 41, you immediately pronounce that 41 is the number thought of. And thus it will always happen in every case.

To construct and fill up these columns of numbers: after having entered the regular series 1, 2, 4, 8, 16, 32, as the top numbers of the six columns in their order; then all the other numbers, in each column downwards, are generated and produced from the number that has been entered at its top, as the first number of that column, by one easy, simple, and general rule, which is this. To the first number, add successively a unit as often as one less than that number denotes, to produce as many of the following numbers; then, to the last of these, add at once a number which is one more than the top number, for the next succeeding number. Repeat the same operations over and over again, as far as required; that is, to the last found number add successive units as often as before, and then again at once a number exceeding the top number by 1: and so on continually for the successive descending numbers. In short, if n denote any top number, add 1 as often as $n-1$ times, after which add at once the number $n+1$; after which repeat the same process over and over to the end. Thus, for example, to fill up the 3rd column, the top figure of which is 4, the value of n in this case; therefore add a unit $n-1$, or three times, which gives the next three numbers in the column, 5, 6, 7; to this last add at once $n+1$ or 5, which gives 12 for the next number; after this add again 1 three times, giving 13, 14, 15, for the next following numbers; and again add the number $n+1$ or 5, making 20; and so on always. In this way the reader will try, by the same rule, to fill up all the columns, which will make the whole process familiar to him.

For greater secrecy, you may cover or withdraw from notice the columns, after the person has informed you which they are that contain his number thought of. And, for the greater mystery, to prevent him from discovering your trick of adding the top numbers together, which he may be apt to do after several repetitions of the trial, you may remove the top numbers lower down in the columns, mixing promiscuously a few of the other numbers at the top.

PROBLEM II.

To tell two or more numbers which a person has thought of.

I.

When each of the numbers thought of does not exceed 9, they may be easily found in the following manner.

Having made the person add 1 to the double of the first number thought of, desire him to multiply the whole by 5, and to add to the product the second number. If there be a third, make him double this first sum, and add 1 to it; after which desire him to multiply the new sum by 5, and to add to it the third number. If there be a fourth, you must proceed in the same manner, desiring him to double the preceding sum; to add to it unity; to multiply by 5, and then to add the fourth number, and so on.

Then ask the number arising from the addition of the last number thought of, and if there were two numbers, subtract 5 from it; if three, 55, if four, 555, and so on; for the remainder will be composed of figures of which the first on the left will be the first number thought of, the next the second, and so of the rest.

Suppose the numbers thought of to be 3, 4, 6: by adding 1 to 6, the double of the first, we have 7, which being multiplied by 5, gives 35; if 4, the second number thought of, be then added, we shall have 39, which doubled gives 78, and if we add 1, and multiply 79, the sum, by 5, the result will be 395. In the last place, if we add 6, the third number thought of, the sum will be 401; and if 55 be deducted from it, we shall have for remainder 346; the figures of which 3, 4, 6, indicate in order the three numbers thought of.

One method we shall here omit, as we shall have occasion to employ it in another amusement of the same kind, called the game of the ring.

II.

If one or more of the numbers thought of are greater than 9, two cases must be distinguished: 1st. that where the number of the numbers thought of is odd. 2d. that where it is even.

In the first case, desire the person to tell the sums of the first and the second; of the second and the third; of the third and the fourth; &c, as far as the last, and then the sum of the first and the last. Having written down these sums in order, add together all those the places of which are odd, as the first, the third, the fifth, &c; make another sum of all those the places of which are even, as the second, the fourth, the sixth, &c; subtract this sum from the former, and the remainder will be the double of the first number.

Let us suppose, for example, that the 5 following numbers are thought of, viz: 3, 7, 13, 17, 20, which, when added two and two, as above, give 10, 20, 30, 37, 23: the sum of the first and third and fifth, is 63; and that of the second and fourth is 57: if 57 be subtracted from 63, the remainder 6 will be the double of the first number 3. Now if 3 be taken from 10, the first of the sums, the remainder 7 will be the second number; and, by proceeding in the same manner, we may find all the rest.

In the second case, that is to say, when the number of the numbers thought of is even; ask, and write down as above, the sum of the first and the second; that of the second and third; and so on as before; but instead of the sum of the first and the last, take that of the second and the last; then add together those which stand in the even places, and form them into a new sum apart: add also those in the odd places, the first excepted, and subtract this sum from the former: the remainder will be the double of the second number; and if the second number thus found be subtracted from the sum of the first and second, the remainder will be the first number; if it be taken from that of the second and third, it will give the third; and so of the rest.

Let the numbers thought of be, for example, 3, 7, 13, 17: the sums formed as above are 10, 20, 30, 24, the sum of the second and fourth is 44, from which if 30 the third sum be subtracted, the remainder will be 14, the double of 7 the second number. The first therefore is 3, the third 13, and the fourth 17.

PROBLEM III.

A person having in one hand an even number of shillings, and in the other an odd, to tell in which hand he has the even number.

Desire the person to multiply the number in the right hand by any even number whatever, such as 2; and that in the left by an odd number, as 3; then bid him add together the two products, and if the whole sum be odd, the even number of shillings will be in the right hand, and the odd number in the left; if the sum be even, the contrary will be the case.

Let us suppose, for example, that the person has 8 shillings in his right hand, and 7 in his left; 8 multiplied by 2 gives 16, and 7 multiplied by 3 gives 21; the sum of which, 37, is an odd number.

If the number in the right hand were 9, and that in the left 8, we should have $9 \times 2 = 18$, and $8 \times 3 = 24$; the sum of which two products is 42, an even number.

PROBLEM IV.

A person having in one hand a piece of gold, and in the other a piece of silver, to tell in which hand he has the gold, and which the silver.

For this purpose, some value, represented by an even number, such as 8, must be assigned to the gold, and a value represented by an odd number, such as 3, must be assigned to the silver: after which the operation is exactly the same as in the preceding example.

REMARKS.

I. To conceal the artifice better, it will be sufficient to ask whether the sum of the two products can be halved without a remainder; for, in that case, the total will be even, and in the contrary case odd.

II. It may be readily seen that the pieces, instead of being in the two hands of the same person, may be supposed to be in the hands of two persons, one of whom has the even number, or piece of gold, and the other the odd number, or piece of silver. The same operations may then be performed in regard to these two persons as are performed in regard to the two hands of the same person, calling the one privately the right, and the other the left.

PROBLEM V.

The Game of the Ring.

This game is nothing else than an application of one of the methods employed to tell several numbers thought of, and should be performed in a company not exceeding 9, in order that it may be less complex. Desire any one of the company to take a ring, and to put it on any joint of whatever finger he may think proper. The question then is to tell what person has the ring, and on what hand, what finger, and what joint.

For this purpose, call the first person 1, the second 2, the third 3, and so on; also call the right hand 1, and the left 2: the first finger of the hand, that is to say the thumb, must be denoted by 1, the second by 2, and so on to the little finger; and the first joint of each finger, or that next the extremity, must be called 1, the second 2, and the third 3.

Let us now suppose that the fifth person has taken the ring, and put it on the first joint of the fourth finger of his left hand. To resolve the problem, nothing is necessary but to discover these numbers 5, 2, 4, 1, which may be done in the following manner.

Desire some one to double the first number 5, which will give 10, and to subtract 1 from it; desire him to multiply 9, the remainder, by 5, which will give 45; to this product bid him add the second number 2, which will make 47, and then 5 which will make 52: desire him to double this number, and the result will be 104, and to subtract 1, which will leave 103. Desire him to multiply this remainder by 5, which will give 515, and to add to the product the third number 4, or that expressing the finger, which will give 519: then bid him add 5, which will make 524, and from 1048, the double of this sum, let him subtract 1, which will leave 1047: then desire him to multiply this remainder by 5, which will give 5235, and to add to this product 1, the fourth number, or that expressing the joint, which will make 5236; in the last place bid him again add 5, and the sum will be 5241, the figures of which will indicate, in order, the person who has the ring, and the hand, finger, and joint, on which it was put.

It is evident, that all these operations amount, in reality, to nothing else than multiplying by 10, the number which expresses the person; then adding that which expresses the hand; multiplying again by 10, and so on*. But as this artifice is too easily detected, it might be better to employ the method taught in Prob. II. No. 1, to discover any number of numbers thought of at pleasure; for, on account of the number which must be subtracted, the operation will be more difficult to be comprehended.

The problem might be proposed in the following manner, and be resolved by the same process.

Three or more persons having each selected a card, the number of the spots of which does not exceed 9, to tell the number of the spots of each.

Desire the first person to add 1 to double the number of the spots of his card; to multiply the sum by 5, and to add to the product the spots of the card of the second person: then desire him to double that sum; to add unity to it, to multiply the whole by 5, and to add to this product the spots of the card of the third person: by subtracting from the last result 55, if the number of the persons be 3; 555, if it be 4: 5555, if it be 5, the figures which compose the remainder will indicate, in order, the spots of the cards selected by each person.

* For the satisfaction and information of the reader, we shall here give the following demonstration.

Let the four numbers to be guessed be x, y, z, u: according to the above method, we must double x, which will give $2x$; if 1 be then subtracted we shall have $2x-1$, and multiplying by 5, the result will be $10x-5$. If y, the second number, be added, we shall have $10x-5+y$, and 5 added to this sum will make $10x+y$, which being doubled will give $20x+2y$; if 1 be subtracted, there will remain $20x+2y-1$, which multiplied by 5 will give $100x+10y-5$; to this product if the third number z, and 5 be added, the sum will be $100x+10y+z$; and if unity be taken from the double of this sum, the result will be $200x+20y+2z-1$; if we then multiply by 5, we shall have for product $1000x+100y+10z-5$; and by adding 5 and the last number, u, the sum will be $1000x+100y+10z+u$. If x, y, z, u represent numbers below 10, as 5, 2, 4, 1, the sum will be $5000+200+40+1$ or 5241. If the numbers were 9, 6, 5, 4, the sum for the same reason would be 9654; which is a demonstration of the process above indicated.

This process may be demonstrated with as much ease as the former; let the numbers to be guessed, less than 10, be x, y, z: we confine ourselves to three, for the sake of brevity. If 1 be added to the double of the first number, we shall have $2x + 1$, and multiplying by 5, the product will be $10x + 5$; if the second number y be added, the sum will be $10x + 5 + y$, and 1 added to the double will make $20x + 10 + 2y + 1$, which multiplied by 5 gives $100x + 50 + 10y + 5$; if we then add the third number z, we shall have $100x + 50 + 10y + 5 + z$ or $100x + 10y + z + 55$: if x, y, z are, for example, 5, 6, 7, this expression will be $567 + 55$ or 612. From this last sum therefore, if we deduct 55, the remainder will be 567, which indicates in order the three numbers to be guessed.

For the sake of brevity, we shall not give any other example, as the reader may recur to that before given in Prob. II.

PROBLEM VI.

To guess the number of spots on any card, which a person has drawn from a whole pack.

Take a whole pack, consisting of 52 cards, and desire some person in company to draw out any one at pleasure, without showing it. Having assigned to the different cards their usual value, according to their spots, call the knave 11, the queen 12, and the king 13. Then add the spots of the first card to those of the second; the last sum to the spots of the third, and so on, always rejecting 13, and keeping the remainder to add to the following card. It may be readily seen that it is needless to reckon the kings, which are counted 13. If any spots remain at the last card, subtract them from 13, and the remainder will indicate the spots of the card that has been drawn; if the remainder be 11, it has been a knave; if 12 it has been a queen, but if nothing remains, it has been a king. The colour of the king may be known by examining which one among the cards is wanting.

If you are desirous of employing only 32 cards, the number used at present for piquet, when the cards are added as above directed, reject all the tens; then add 4 to the spots of the last card, and a sum will be obtained which taken from 10, if it be less, or from 20 if it exceeds 10, the remainder will be the number of the card that has been drawn; so that if 2 remains, it has been a knave, if 3 a queen, if 4 a king, and so on.

If the pack be incomplete, attention must be paid to those deficient, in order that the number of the spots of all the cards wanting may be added to the last sum, after as many tens as possible have been subtracted from it; and the sum arising from this addition, must, as before, be taken from 10 or 20 according as it is greater or less than 10. It is evident that by again looking at the cards, the one which has been drawn may be discovered.

The demonstration of this rule is as follows: since, in a complete pack of cards, there are 13 of each suit, the values of which are 1, 2, 3, &c, to 13, the sum of all the spots of each suit, calling the knave 11, the queen 12, and the king 13, is seven times 13, or 91, which is a multiple of 13; consequently the quadruple of this sum is a multiple of 13 also: if the spots then of all the cards be added together, always rejecting 13, we must at last find the remainder equal to nothing. It is therefore evident that if a card, the spots of which are less than 13, has been drawn from the pack, the difference between these spots and 13, will be what is wanting to complete that number: if at the end then, instead of reaching 13, we reach only 10, for example, it is evident that the card wanting is a three; and if we reach 13, it is also evident that the card wanting is one of those equivalent to 13, or a king.

If two cards have been drawn from the pack, we may tell, in like manner, the number of spots which they contain both together: that is, how much is wanting to reach 13, or that deficiency increased by 13; and to know which two, nothing is necessary but to count privately how many times 13 has been completed, for with the whole of the cards it ought to be counted 28 times: if it be counted therefore only 27 times, with a remainder, as 7 for example, the spots of the two cards drawn amount together to 6: if 13 be counted only 26 times, with the same remainder, it may be concluded that the two cards form together 13 + 6, or 19.

The demonstration of the rule given when the same number of cards is used, as that employed for the game of piquet, viz. 32, calling the ace 1, the knave 2, the queen 3, the king 4, and assigning to the other cards the value of their spots, is attended with as little difficulty; for in each suit there are 44 spots, making all together 176, which, as well as 44, is a multiple of 11, we may therefore always count to 11, rejecting 11, and the number wanting to reach 11, will be the value of the card which has been drawn.

But the same number 176, if 4 were added to it, would be a multiple of 10 or of 20; and hence a demonstration also of the method which has been taught.

PROBLEM VII.

A person having an equal number of counters, or pieces of money, in each hand, to find how many he has altogether.

Desire the person to convey any number, as 4, for example, from the one hand to the other, and then ask him how many times the less number is contained in the greater. Let us suppose that he says the one is triple of the other; in this case multiply 4, the number of the counters conveyed from one hand into the other, by 3, and add to the product, the same number 4, which will make 16. In the last place, from the number 3 subtract unity, and if 16 be divided by 2, the remainder, the quotient 8 will be the number contained in each hand, and consequently the whole number is 16.

Let us now suppose that when 4 counters are conveyed from one hand to the other, the less number is contained in the greater $2\frac{1}{3}$ times: in this case we must, as before, multiply 4 by $2\frac{1}{3}$, which will give $9\frac{1}{3}$; to which if 4 be added, we shall have $13\frac{1}{3}$, or $\frac{40}{3}$; if unity be then taken from $2\frac{1}{3}$, the remainder will be $1\frac{1}{3}$, or $\frac{4}{3}$, by which if $\frac{40}{3}$ be divided, the quotient, 10, will be the number of counters in each hand, as may be easily proved on trial.

PROBLEM VIII.

Several cards being shown to a person, to tell that which he has thought of.

Having taken any number of cards at pleasure, from a whole pack, display them in order on the table, that the person may choose one. When this is done, place them carefully one above the other, beginning with the lower one, and desire the person to remember the number expressing the order of the card he thought of, viz. 1, if he thought of the first, 2 if of the second, and so on; but at the same time, count privately the number of cards shewn to the person, which we shall suppose to be 12, and separate them dexterously from the remainder of the pack. Then place these cards on the remainder of the pack in an inverted position, beginning with that first displayed on the table, and ending with that which was last. Having then asked the number of the card thought of, which we shall suppose to be the fourth, lay your cards on the table with their faces uppermost, one after the other, beginning with that at the top, to which you must assign 4, the number of the card thought of, calling the next card which follows, or the second, 5, the third 6, and so on, until you come to 12, the number of the cards you at first assumed; for the card on which that number falls will be the card thought of.

PROBLEM IX.

Several cards being presented, in succession, to several persons, that they may each choose one at pleasure; to guess that which each has thought of.

Shew as many cards to each person as there are persons to choose; that is to say 3 to each if there are 3 persons. When the first has thought of one, lay aside the three cards in which he has made his choice. Present the same number to the second person, to think of one, and lay aside the three cards in the like manner. Having done the same in regard to the third person, spread out the three first cards with their faces upwards, and place above them the next three cards, and above these the last three, that all the cards may thus be disposed in three heaps, each consisting of three cards. Then ask each person in which heap the card is which he thought of, and when this is known it will be easy to tell these cards, for that of the first person will be the first in the heap to which it belongs; that of the second will be the second of the next heap, and that of the third will be the third of the last heap.

PROBLEM X.

Three cards being presented to three persons, to guess that which each has chosen.

As it is necessary that the cards presented to the three persons should be distinguished, we shall call the first A, the second B, and the third C; but the three persons may be at liberty to choose any of them at pleasure. This choice, which is susceptible of six different varieties, having been made, give to the first person 12 counters, to the second 24, and to the third 36: then desire the first person to add together the half of the counters of the person who has chosen the card A, the third of those of the person who has chosen B, and the fourth part of those of the person who has chosen c, and ask the sum, which must be either 23 or 24; 25 or 27; 28 or 29, as in the following table:

First	Second	Third	Sums
12	24	36	
A	B	C	23
A	C	B	24
B	A	C	25
C	A	B	27
B	C	A	28
C	B	A	29

This table shows, that if the sum be 25, for example, the first person must have chosen the card B, the second the card A, and the third the card C; and that if it be 28, the first person must have chosen the card B, the second the card C, and the third the card A; and so of the rest.

PROBLEM XI.

A person having drawn, from a complete pack of fifty-two cards, one, two, three, four, or more cards, to guess the whole number of the spots which they contain.

Assume any number whatever, such as 15, for example, greater than the number of the spots of the highest card, counting the knave 11, the queen 12, and the king 13; and desire the person to add as many cards from the pack, to the first card he has chosen, as will make up 15, counting the spots of that card; let him do the same thing in regard to the second, the third, the fourth, &c; and then desire him to tell how many cards remain in the pack.— When this is done, proceed as follows:

Multiply the above number 15, or any other that may have been assumed, by the number of cards drawn from the pack, which we shall here suppose to be 3; to the product, 45, add the number of these cards, which will give 48; subtract the 48 from 52, and take the remainder 4 from the cards left in the pack: the result will be the number of spots required.

Let us suppose, for example, that the person has drawn from the pack a 7, a 10, and a knave, which is equal to 11: to make up the number 15 with a 7, eight cards will be required; to make up the same number with a 10, will require five; and with the knave, which is equal to 11, four will be necessary. The sum of these three numbers, with the 3 cards, makes 20, and consequently 32 cards remain in the pack. To find the sum of the numbers 7, 10, 11, multiply 15 by 3, which will give 45; and if the number of the cards drawn from the pack be added, the sum will be 48, which taken from 52, leaves 4. If 4 then be subtracted from 32, the remainder, 28, will be the sum of the spots contained on the three cards, drawn from the pack, as may be easily proved by trial.

Another Example.

Let us suppose two cards only drawn from the pack, a 4 and a king, equal to 13; if cards be added to these to make up 15, there will remain in the pack 37 cards.

If 15 be multiplied by 2, the product will be 30, to which if 2, the number of the cards drawn from the pack, be added, we shall have 32; and if 32 be taken from 52, the remainder will be 20. In the last place, if 20 be subtracted from 37, the number of the cards left in the pack, the remainder, 17, will be the number of the spots of the 2 cards drawn from the pack.

REMARKS.

I. If 4 or 5 cards are drawn from the pack, it may sometimes happen that a sufficient number will not be left to make up the number 15; but even in this case the operation may be still performed. For example, if 5 cards, the spots contained on which are 1, 2, 3, 4, 5, have been drawn; to complete with each of these cards the number

15 would require, together with the 5 cards, at least 65; but as there are only 52, there are consequently 13 too few. He who counts the pack must therefore say that 13 are wanting.

On the other hand, he who undertakes to tell the number of the spots, must multiply 15 by 5, which makes 75; and to this if 5, the number of the cards, be added, it will give 80; that is to say, 28 more than 52: if 13 then be subtracted from 28, the remainder 15 will be the number of the spots contained on these 5 cards.

But if we suppose that the cards left in the pack are, for example, 22, which would be the case if the five cards drawn were the 8, 9, 10, knave = 11, and queen = 12, it would be necessary to add these 22 to the excess of 5 times 15 + 5, over 52, that is to say to 28, and we should have 50 for the spots of these 5 cards, which is indeed the exact number of them.

II. If the pack consists not of 52 cards, but of 40, for example, there will still be no difference in the operation: the number of the cards, which remain of these 40, must be taken from the sum produced by multiplying the made up number, by that of the cards drawn, and adding to the product the number of these cards.

Let us suppose, for example, that the cards drawn are 9, 10, 11, that the number to be made up is 12, and that the cards left in the pack are 31. Then $12 \times 3 = 36$, and 3 added for the 3 cards, makes 39, which subtracted from 40 leaves 1. If 1 then be taken from 31, the remainder 30 will be the number of the spots required.

III. Different numbers to be made up with the spots of each card chosen might be assumed; but the case would still be the same, only that it would be necessary to add these three numbers to that of the cards, instead of multiplying the same number by the number of cards drawn, and then adding the number of the cards. In this there is so little difficulty, that an example is not necessary.

IV. The demonstration of this method, which some of our readers perhaps may be desirous of seeing, is exceedingly simple, and is as follows. Let *a* be the number of cards in the pack, *c* the number to be made up by adding cards to the spots of each card drawn, and *b* the cards left in the pack; let *x*, *y*, *z* express the spots of the cards, which we shall here suppose to be 3, and we shall then have, for the number of the cards drawn, $c-x+c-y+c-z+3$; which with the cards left in the pack *b*, must be equal to the whole pack. Then $3c+3-x-y-z+b=a$, or $x+y+z=3c+3+b-a$ or $=b-(a-3c-3)$. But $x+y+z$ is the whole number of the spots; *b* is the number of cards left in the pack, and $a-3c-3$ is the whole number of cards in the pack, less the product of the number to be completed by the number of the cards drawn, minus that number. Therefore, &c.

PROBLEM XII.

Three things being privately distributed to three persons; to guess that which each has got.

Let the three things be a ring, a shilling, and a glove. Call the ring A, the shilling E, and the glove I; and in your own mind distinguish the persons by calling them first, second, and third. Then take twenty-four counters, and give one of them to the first person, two to the second, and three to the third. Place the remaining 18 on the table, and then retire, that the three persons may distribute among themselves the three things proposed, without your observing them. When the distribution has been made, desire the person who has the ring to take from the 18 remaining counters as many as he has already; the one who has the shilling to take twice as many as he has already, and the person who has the glove to take four times as many; according to the above supposition then, the first person has taken 1, the second 4, and the third 12; consequently one counter only remains on the table. When this is done, you may return, and by the number left can discover what thing each has got, by employing the following words:

1 2 3 5 6 7

Par fer César jadis devint si grand prince.

To make use of these words, you must recollect, that in all cases there can remain only 1 counter, or 2, 3, 5, 6, or 7, and never 4: it must be likewise observed that each syllable contains one of the vowels, which we have made to represent the three things proposed, and that the above line must be considered as consisting only of six words: the first syllable of each word must also be supposed to represent the first person, and the second syllable the second. This being comprehended, if there remains only 1 counter, you must employ the first word, or rather the two first syllables, *par fer*, the first of which, that containing A, shews that the first person has the ring represented by A; and the second syllable, that containing E, shews that the second person has the shilling, represented by E; from which you may easily conclude that the third person has the glove. If two counters remain, you must take the second word *César*, the first syllable of which, containing E, will shew that the first person has the shilling, represented by E, and the second syllable, containing A, will indicate that the second person has the ring, represented by A: you may then easily conclude that the third person has the glove. In general, whatever number of counters remain, that word of the verse, which is pointed out by the same number, must be employed.

REMARKS.

Instead of the above French verse, the following Latin one might be used:

1 2 3 5 6 7

Salve certa animæ semita vita quies.

This problem might be proposed in a manner somewhat different, and might be applied to more than three persons: those who are desirous of farther information on the subject, may consult Bachet in the 25th of his *Problêmes plaisant et delectables.*

PROBLEM XIII.

Several numbers being disposed in a circular form, according to their natural series, to tell that which any one has thought of.

The first ten cards of any suit, disposed in a circular form, as seen in the figure below, may be employed with great convenience for performing what is announced in this problem. The ace is here represented by the letter A annexed to 1, and the ten by the letter K joined to 10.

	2	3	4	
	B	C	D	
1 A				E 5
10 K				F 6
	I	H	G	
	9	8	7	

Having desired the person, who has thought of a number or card, to touch also any other number or card, bid him add to the number of the card touched, the number of the cards employed, which in this case is 10. Then desire him to count that sum in an order contrary to that of the natural numbers, beginning at the card he touched, and assigning to that card the number of the one which he thought of; for by counting in this manner, he will end at the number or card which he thought of, and consequently you will easily know it.

Thus, for example, if the person has thought of the number 3, marked C, and has touched 6, marked F; if 10 be added to 6, it will make 16; if 16 be then counted* from F, the number touched, towards E, D, C, B, A, and so

* It is to be observed that the person must not count this sum aloud, but privately in his own mind.

on in the retrograde order, counting 3, the number thought of, on F, 4 on E, 5 on D, 6 on C, and so round to 16, the number 16 will terminate on C, and show that the person thought of 3, which corresponds to C.

REMARKS.

I. A greater or less number of cards may be employed at pleasure. If there are 15 or 8 cards, 15 or 8 must be added to the number of the card touched.

II. To conceal the artifice better, you may invert the cards, so as to prevent the spots from being seen, but you must remember the natural series of the cards, and the place of the first number, or the ace, that you may know the number of the card touched, in order to find the one to which the person ought to count.

PROBLEM XIV.

Two persons agree to take alternately numbers less than a given number, for example 11, *and to add them together till one of them has reached a certain sum, such as* 100; *by what means can one of them infallibly attain to that number before the other?*

The whole artifice of this problem consists in immediately making choice of certain numbers, which we shall here point out. Subtract 11, for example, from 100, the number to be reached, as many times as possible, and the remainders will be 89, 78, 67, 56, 45, 34, 23, 12, and 1, which must be remembered; for he who by adding his number less than 11, to the sum of the preceding, shall count one of these numbers before his adversary, will infallibly win, without the other being able to prevent him.

These numbers may be found also, with still greater ease, by dividing 100 by 11, and adding 11 continually to 1, the remainder, which will give 1, 12, 23, 34, &c.

Let us suppose, for example, that the first person, who knows the game, takes 1 for his number: it is evident that his adversary, as he must count less than 11, can at most reach 11 by adding 10 to it. The first will then take 1, which will make 12; if the second takes 8, which will make 20, the first will take 3, which will make 23; and proceeding in this manner successively he will first reach 34, 45, 56, 67, 78, 89. When he attains to the last number, it will be impossible for the second to prevent him from getting first to 100; for whatever number the second takes, he can attain only to 99, after which the first may say, "and 1 makes 100." If the second takes 1 after 89, it will make 90; and his adversary may finish by saying, "and 10 make 100."

It is evident that when two persons are equally well acquainted with the game, he who begins must necessarily win.

But, if the one knows the game, and the other does not, the latter, though first, may not win; for he will think it highly advantageous to take the greatest number possible, that is to say 10; and in that case the other, acquainted with the nicety of the game, will take 2, which with 10 will make 12, one of the numbers he ought to secure. But he may even neglect this advantage, and take only 1 to make 11; for the first will probably still take 10, which will make 21, and the second may then take 2, which will make 23; he may then wait a little longer to get hold of some of the following numbers 34, 45, 56, &c.

If the first is desirous to win, the least number proposed must not be a measure of the greater; for in that case the first would have no infallible rule to direct him in his operations. For example, if 10, which measures 100, were assumed, instead of 11, by subtracting 10 from 100 as many times as possible, we should have the numbers 10, 20, 30, 40, 50, 60, 70, 80, 90, the first of which, 10, could not be taken by the first; for being obliged to employ a number less than 10, if the second were as well acquainted with the game, he might take the complement to 10; and would thus have an infallible rule for winning.

PROBLEM XV.

Sixteen counters being disposed in two rows, to find that which a person has thought of.

The counters being arranged in two rows, as A and B, desire the person to think of one, and to observe well in which row it is.

A	B	C	B	D	E	B	F	H	B	I
o	o	o	o	o	o	o	o	o	o	*
o	o	o	o	o	*	o	o	o	o	o
o	o	*	o	o	o	o	o	o	o	o
o	o	o	o	o	o	o	o	o	o	o
*	o		o			o			o	
o	o		o			o			o	
o	o		o			o			o	
o	o		o			o			o	

Let us suppose that the counter thought of, is in the row A; take up that whole row, in the order in which it stands, and dispose it in two rows C and D, on the right and left of the row B, but in arranging them, take care that the first of the row A may be the first of the row C; the second of the row A, the first of the row D; the third of the row A, the second of the row C, and so on; then ask again in which of the vertical rows, C or D, the counter thought of is. Suppose it to be in C: take up that row as well as the row D, putting the last at the end of the first, without deranging the order of the counters, and, observing the rule already given, form them into two other rows, as seen at E and F; then ask, as before, in which row the counter thought of is. Let us suppose it to be in E; take up this row, and the row F, as above directed, and form them into two new rows, on the right and left of

B. After these operations, the counter thought of must be the first of one of the perpendicular rows, H and I; if you therefore ask in which row it is, you may easily point it out; and as it is here supposed that each of the counters has some distinguishing mark, you may desire them to be mixed together, and still be able to tell the one thought of, by observing the mark.

It may be readily seen that, instead of counters, sixteen cards may be employed; and when you have discovered, by the above means, the one thought of, you may cause them to be mixed, which will better conceal the artifice.

REMARK.

If a greater number of counters or cards, arranged in two vertical rows, be supposed, the counter or card thought of will not necessarily be the first in the row to which it belongs, after the third transposition: if there be 32 counters or cards, four transpositions will be necessary; if there are 64, five; and so on, before it can be said, with confidence, that the counter or card thought of occupies the first place in its row; for if this counter or card were at the bottom of the perpendicular row A, supposing 16 counters in each row, or 32 altogether, it would not arrive at the first place till after four transpositions: if there were 64, or 32 in each row, it would require five; and so on, as may be easily proved by trial.

PROBLEM XVI.

A certain number of cards being shown to a person, to guess that which he has thought of.

To perform this trick, the number of the cards must be divisible by 3; and to do it with more convenience, the number must be odd.

The first condition, at least, being supposed, desire the person to think of a card: then place the cards on the table with their faces downward; and, taking them up in order, arrange them in three heaps, with their faces upward, and in such a manner, that the first card of the packet shall be the first of the first heap; the second the first of the second, and the third the first of the third; the fourth, the second of the first, and so on. When the heaps are completed, ask the person in which heap is the card thought of, and when told, place that containing the card thought of in the middle; then turning up the packet, form three heaps, as before, and again ask in which is the card thought of. Place the heap containing the card thought of still in the middle, and, having formed three new heaps, ask which of them contains the card thought of. When this is known, place it as before between the other two; and again form three heaps, asking the same question. Then take up the heaps for the last time; put that containing the card thought of in the middle, and placing the packet on the table, with the faces of the cards downward, turn up the cards till you count half the number of those contained in the packet; 12 for example if there be 24, in which case the 12th card will be the one the person thought of.

If the number of the cards be, at the same time, odd, and divisible by 3, as 15, 21, 27, &c, the trick will become much easier; for the card thought of will always be that in the middle of the heap in which it is found the third time; so that it may be easily distinguished without counting the cards; nothing will be necessary for this purpose, but to remember, while you are forming the heaps the third time, the card which is the middle one of each. Let us suppose, for example, that the middle card of the first heap is the ace of hearts; that of the second the king of hearts, and that of the third the knave of spades; it is evident, if you are told that the heap containing the required card is the third, that this card must be the knave of spades. You may therefore cause the cards to be shuffled, without touching them any more, and then, looking them over for the sake of form, may name the knave of spades when it occurs.

PROBLEM XVII.

Fifteen Christians and fifteen Turks being at sea in the same vessel, a dreadful storm came on, which obliged them to throw all their merchandise overboard; this however not being sufficient to lighten the ship, the captain informed them that there was no possibility of its being saved, unless half the passengers were thrown overboard also. Having therefore caused them all to arrange themselves in a row, by counting from 9 to 9, and throwing every ninth person into the sea, beginning again at the first of the row when it had been counted to the end, it was found that after fifteen persons had been thrown overboard, the fifteen Christians remained. How did the captain arrange these thirty persons so as to save the Christians?

The method of arranging the thirty persons may be deduced from these two French verses:

Mort, tu ne failliras pas
En me livrant le trepas.

Or from the following Latin one, which is not so bad of its kind:

Populeam virgam mater regina ferebat.

Attention must be paid to the vowels A, E, I, O, U, contained in the syllables of these verses; observing that A is equal to 1, E to 2, I to 3, O to 4, and U to 5. You must begin then by arranging 4 Christians together, because the vowel in the first syllable is O; then 5 Turks, because the vowel in the second syllable is U; and so on to the end. By proceeding in this manner, it will be found, taking every ninth person circularly, that is to say, beginning at the first of the row, after it is ended, that the lot will fall entirely on the Turks.

The solution of this problem may be easily extended still farther. Let it be required, for example, to make the lot fall upon 10 persons in 40, counting from 12 to 12. Arrange 40 ciphers in a circular form, as below;

```
                † †   †   †
        0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
      0                                 0
  1  0                                   0
     0                                   0†
      0                                 0†
        0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
            † † †                   †
```

then, beginning at the first, mark every 12th one with a cross; continue in this manner, taking care to pass over those already crossed, still proceeding circularly, till the required number of places has been marked; if you then count the places of the marked ciphers, those on which the lot falls will be easily known: in the present case they are the 7th, the 8th, the 10th, the 12th, the 21st, the 22d, the 24th, the 34th, the 35th, and the 36th.

A captain, obliged to decimate his company, might employ this expedient, to make the lot fall upon those most culpable.

It is related that Josephus, the historian, saved his life by means of this expedient. Having fled for shelter to a cavern, with forty other Jews, after Jotapat had been taken by the Romans, his companions resolved to kill each other rather than surrender. Josephus tried to dissuade them from their horrid purpose, but not being able to succeed, he pretended to coincide with their wishes, and retaining the authority he had over them as their chief, to avoid the disorder which would necessarily be the consequence of this cruel execution, if they should kill each other at random, he prevailed on them to arrange themselves in order, and, beginning to count from one end to a certain number, to put to death the person on whom that number should fall, until there remained only one, who should kill himself. Having all agreed to this proposal, Josephus arranged them in such a manner, and placed himself in such a position, that when the slaughter had been continued to the end, he remained with only one more person, whom he persuaded to live.

Such is the story related of Josephus by Hegesippus; but we are far from warranting the truth of it. However, by applying to this case the method above indicated, and supposing that every third person was to be killed, it will be found that the two last places on which the lot fell were the 16th and 31st; so that Josephus must have placed himself in one of these, and the person he was desirous of saving, in the other.

PROBLEM XVIII.

A man has a wolf, a goat, and a cabbage, to carry over a river; but being obliged to transport them one by one, on account of the smallness of the boat, in what manner is this to be done, that the wolf may not be left with the goat, nor the goat with the cabbage?

He must first carry over the goat, and then return for the wolf; when he carries over the wolf, he must take back with him the goat, and leave it, in order to carry over the cabbage; he may then return, and carry over the goat. By these means, the wolf will never be left with the goat, nor the goat with the cabbage, but when the boatman is present.

PROBLEM XIX.

Three jealous husbands, with their wives, having to cross a river at a ferry, find a boat without a boatman; but the boat is so small that it can contain no more than two of them at once. How can these six persons cross the river, two and two, so that none of the women shall be left in company with any of the men, unless when her husband is present?

The solution of this problem is contained in the two following Latin distichs:

It duplex mulier, redit una, vehitque manentem,
Itque una; utuntur tunc duo puppe viri.
Par vadit et redeunt bini, mulierque sororem
Advehit; ad propriam fine maritus abit.

That is: "two women cross first, and one of them, rowing back the boat, carries over the third woman. One of the three women then returns with the boat, and remaining, suffers the two men, whose wives have crossed, to go over in the boat. One of the men then carries back his wife, and leaving her on the bank, rows over the third man. In the last place, the woman who had crossed enters the boat, and returning twice, carries over the other two women."

This question is proposed also under the title of the three masters and the three valets. The masters agree very well, and the valets also; but none of the masters can endure the valets of the other two; so that if any one of them were left with any of the other two valets, in the absence of his master, he would infallibly cane him.

PROBLEM XX.

In what manner can counters be disposed in the eight external cells of a square, so that there may be always 9 *in each row, and yet the whole number shall vary from* 20 *to* 32?

Ozanam proposed this problem in the following manner, with a view no doubt to excite the curiosity of his readers:

A certain convent consisted of nine cells, one of which in the middle was occupied by a blind abbess, and the rest by her nuns. The good abbess, to assure herself that the nuns did not violate their vows, visited all the cells, and finding 3 nuns in each, which made 9 in every row, retired to rest. Four nuns however went out, and the abbess returning at midnight to count them, still found 9 in each row, and therefore retired as before. The four nuns then came back, each with a gallant, and the abbess on paying them another visit, having again counted 9 persons in each row, entertained no suspicion of what had taken place. But 4 more men were introduced, and the abbess again counting 9 persons in each row, retired in the full persuasion that no one had either gone out, or come in. How was all this possible?

This problem may be easily solved by inspecting the four following figures; the first of which represents the original disposition of the counters in the cells of the square; the second that of the same counters when 4 are taken away; the third the manner in which they must be disposed when these 4 are brought back with 4 others; and the fourth that of the same counters with the addition of 4 more. It is here evident that there are always 9 in each external row; and yet, in the first case, the whole number is 24, while in the second it is 20, in the third 28, and in the fourth 32.

I.

3	3	3
3		3
3	3	3

II.

4	1	4
1		1
4	1	4

III.

2	5	2
5		5
2	5	2

IV.

1	7	1
7		7
1	7	1

It would seem that Ozanam had not observed that these variations might have been carried still farther: that 4 men more might have been introduced into the convent, without the abbess perceiving it; and that all the men might have afterwards gone out with six nuns, so as to leave only 18, instead of the 24 who were in the cells at first. The possibility of this will appear by inspecting the two following figures.

V.

0	9	0
9		9
0	9	0

VI.

5	0	4
0		0
4	0	5

It is almost needless to explain in what manner the illusion of the good abbess arose. It is because the numbers in the angular cells of the square were counted twice; these cells being common to two rows. The more, therefore, the angular cells are filled, by emptying those in the middle of each band, these double enumerations become greater; on which account the number, though diminished, appears always to be the same; and the contrary is the case in proportion as the middle cells are filled, by emptying the angular ones, which renders it necessary to add some units to have 9 in each band.

PROBLEM XXI.

A gentleman has a bottle, containing 8 pints, of choice wine, and wishes to make a present of one half of it to a friend; but as he has nothing to measure it, except two other bottles, one capable of containing 5 and the other 3 pints, how must he manage, so as to put exactly 4 pints into the bottle capable of containing 5?

To enable us to resolve this problem, we shall call the bottle containing the 8 pints, A; that of 5 pints, B; and that of 3 pints, C; supposing that there are 8 pints of wine in the bottle A, and that the other two are empty, as seen at D. Having filled the bottle B, with wine from the bottle A, in which there will remain no more than 3 pints, as seen at E, fill the bottle C from B, and consequently there will remain only 2 pints in the latter, as seen at F: then pour the wine of C into A, which will thus contain 6 pints, as seen at G, and pour the two pints of B into C, as seen at H. In the last place,

	8	5	3
	A	B	C
D	8	0	0
E	3	5	0
F	3	2	3
G	6	2	0
H	6	0	2
I	1	5	2
K	1	4	3

having filled the bottle B from the bottle A, in which there will remain only 1 pint, as seen at I, fill up C from B, in which there will remain 4 pints, as seen at K; and thus the problem is solved.

REMARK.

If you are desirous of making the four pints of wine remain in the bottle A, which we have supposed to be filled with 8 pints, instead of remaining in the bottle B, fill the bottle C with wine from the bottle A, in which there will remain only 5 pints, as seen at D; and pour the 3 pints of C into B, which will consequently contain 3 pints, as seen at E: having then filled C from A, in which there will remain no more than 2 pints, as seen at F; fill up B from C, which will thus contain only 1 pint, as seen at G. In the last place, having poured the wine of the bottle B into the bottle A, which will thus have 7 pints, as seen at H; pour the pint of wine which is in C into B, consequently the latter will contain 1 pint, as seen at I; and then fill up C from A, in which there will remain only 4 pints, as was proposed, and as seen at K.

	8	5	3
	A	B	C
	8	0	0
D	5	0	3
E	5	3	0
F	2	3	3
G	2	5	1
H	7	0	1
I	7	1	0
K	4	1	3

PROBLEM XXII.

A gentleman has a bottle containing 12 pints of wine, six of which he is desirous of giving to a friend; but as he has nothing to measure it, except two other bottles, one of 7 pints, and the other of 5, how must he manage, to have the 6 pints in the bottle capable of containing 7 pints?

This problem is of the same nature as the preceding, and may be solved in the like manner. Let T represent the twelve-pint, S the seven-pint, and F the five-pint bottle. The bottle T is full, and the other two, S and F, are empty, as seen at G. Fill the bottle F with wine from T, so that T shall contain only 7 pints, as seen at H; then pour into S the wine contained in F, which will remain empty, and the bottle S will contain 5 pints, as seen at I; having filled F from T, the latter will contain only 2 pints, the bottle S will contain 5, and the bottle F will be full, as seen at K: in the next place fill S from F, and T will still contain only 2 pints, while S contains 7, and F 3, as seen at L; then empty S into T, and F into S, by which means T will contain 9 pints, and S 3, F remaining empty, as seen at M: fill F from T, and pour from F into S as much as will fill it, so that there will then be 4 pints in T, 7 pints in S, and 1 pint in F, as seen at N: pour the 7 pints from S into T, and the pint contained in F into S, after which T will contain 11 pints, S 1, and F will be empty, as seen at O. In the last place, having filled the five-pint bottle F from the bottle T, and poured these 5 pints from F into S, which already contains 1, it will be found that T contains 6 pints, and that S contains 6 also; so that the desired result has been obtained.

	12	7	5
	T	S	F
G	12	0	0
H	7	0	5
I	7	5	0
K	2	5	5
L	2	7	3
M	9	3	0
N	4	7	1
O	11	1	0
P	6	6	0

PROBLEM XXIII.

To make the knight move into all the squares of the chess board, in succession, without passing twice over the same.

As the reader perhaps may be unacquainted with the movement of the knight at the game of chess, we shall here explain it: The knight, being placed in the square A, cannot move into any of those immediately surrounding it, as, 1, 2, 3, 4, 5, 6, 7, 8; nor into the squares 9, 10, 11, 12, which are directly above or below, and on each side of it; nor into the squares 13, 14, 15, 16, which are in the diagonals; but only into one of those which, in the annexed figure, are empty.

13		10		14
	1	2	3	
9	8	A	4	11
	7	6	5	
16		12		15

Several eminent men have amused themselves with this problem, such as Montmort, Demoivre, and Mairan, and each of these has given a solution of it. In those of the two former, the knight is supposed to be placed at first in one of the angular squares of the chess-board; in that of the third, he is supposed to begin to move from one of the four central squares; but in our opinion it was not known, till within these few years, that placing the knight in any square whatever, he may be made to traverse the whole chess-board, and even in such a manner that, without returning the same way, he shall pass a second time over the board under the like conditions. For this last solution we are indebted to M. W——, a captain in the Kinski regiment of dragoons, in the imperial service.

We shall here give four tables, representing these four solutions, with an explanation and some remarks.

I. *M. Montmort.*

1	38	31	44	3	46	29	42
32	35	2	39	30	43	4	47
37	8	33	26	45	6	41	28
34	25	36	7	40	27	48	5
9	60	17	56	11	52	19	50
24	57	10	63	18	49	12	53
61	16	59	22	55	14	51	20
58	23	62	15	64	21	54	13

II. *Demoivre.*

34	49	22	11	36	39	24	1
21	10	35	50	23	12	37	40
48	33	62	57	38	25	2	13
9	20	51	54	63	60	41	26
32	47	58	61	56	53	14	3
19	8	55	52	59	64	27	42
46	31	6	17	44	29	4	15
7	18	45	30	5	16	43	28

III. *M. Mairan.*

40	9	26	53	42	7	64	29
25	52	41	8	27	30	43	6
10	39	24	57	54	63	28	31
23	56	51	60	1	44	5	62
50	11	38	55	58	61	32	45
37	22	59	48	19	2	15	4
12	49	20	35	14	17	46	33
21	36	13	18	47	34	3	16

IV. *M. W——.*

25	22	37	8	35	20	47	6
38	9	24	21	52	7	34	19
23	26	11	36	59	48	5	46
10	39	62	51	56	53	18	33
27	12	55	58	49	60	45	4
40	63	50	61	54	57	32	17
13	28	1	42	15	30	3	44
64	41	14	29	2	43	16	31

Of these four ways of resolving the problem, that of Demoivre is doubtless the easiest to be remembered; for the principle of his method consists in filling up, as much as possible, the two exterior bands, which form as it were a border, and not entering the third, till there is no other method of moving the knight from the place where he is, to one of the two first, a rule which in the clearest manner subjects the movement of the knight to a certain necessary progress, from his first step to the 50th, and beyond it; for from the cell marked 50 there is no choice in placing him, except on those marked 51 and 63; but the cell 51 being nearer the band, ought to be preferred, and then the movement must necessarily be through 52, 53, 54, 55, 56, 57, 58, 59, 60, 61. When he arrives at 61, it is a matter of indifference whether he be placed in the cell marked 64, for he may thence proceed to the last but one 63, and end at 62: or be placed in 62, to proceed to 63, and end at 64. It may therefore be said that the movement of the knight in this solution is almost constrained.

The case is not the same with the fourth, which it is difficult to practise in any other manner than from memory; but it is attended with one very great advantage, which is, that you may begin, as already said, at any cell at pleasure; because the author took the trouble to bring the knight at the conclusion to a place from which he can pass into the first. His movement therefore is in some measure circular, and interminable, by adhering to the condition of not passing twice over the same cell, till after 64 steps.

It may be readily seen, that to make the knight perform this movement without confusion, the cell he has quitted must be marked at each step. For this purpose a counter may be placed in each cell, and removed as the knight passes over it, or, what will be still better, a counter may be placed in each cell when he has passed it.

PROBLEM XXIV.

To distribute among 3 persons, 21 casks of wine, 7 of them full, 7 of them empty, and 7 half full, so that each shall have the same quantity of wine, and the same number of casks.

This problem admits of two solutions, which may be clearly comprehended by means of the two following tables;

Persons		full casks	empty	half full
I.	1st	2	2	3
	2d	2	2	3
	3d	3	3	1
II.	1st	3	3	1
	2d	3	3	1
	3d	1	1	5

It is evident that, in these two combinations, each person will have 7 casks, and $3\frac{1}{2}$ casks of wine.

But it may be easily seen that the whole number of the casks must be divisible by the number of persons, otherwise the thing required would be impossible.

It will be found, in like manner, that if 24 casks were to be divided among three persons, under the same conditions, we should have three different solutions as follows:

Persons		full casks	empty	half full
I.	1st	3	3	2
	2d	3	3	2
	3d	2	2	4
II.	1st	2	2	4
	2d	2	2	4
	3d	4	4	0
III.	1st	1	1	6
	2d	3	3	2
	3d	4	4	0

If there should be 27 casks to be divided, there would be three solutions also:

	Persons	full casks	empty	half full
I.	1st	3	3	3
	2d	3	3	3
	3d	3	3	3
II.	1st	1	1	7
	2d	4	4	1
	3d	4	4	1
III.	1st	2	2	5
	2d	3	3	3
	3d	4	4	1

CHAPTER XI.

Containing some curious Arithmetical Problems.

PROBLEM I.

A gentleman, in his will, gave orders that his property should be divided among his children in the following manner: the eldest to take from the whole 1000*l. and the 7th part of what remained; the second* 2000*l. and the 7th part of the remainder; the third* 3000*l. and the 7th part of what was left; and so on to the last, always increasing by* 1000*l. The children having followed the disposition of the testator, it was found that they had each got an equal portion: how many children were there, what was the father's property, and to how much did the share of each child amount?*

IT will be found by analysis, that the father's property was 36000*l*; that there were 6 children, and that the share of each was 6000*l*.

Thus, if the first takes 1000*l*. the remainder of the property will be 35000*l*. the 7th part of which 5000*l*. together with 1000*l*. makes 6000*l*. The remainder, after deducting the first child's portion, is 30000*l*. from which if the second takes 2000*l*. the remainder will be 28000*l*. but the 7th part of this sum is 4000*l*. which if added to the above 2000*l*. will make 6000*l*. and so on for the rest.

PROBLEM II.

A gentleman meeting a certain number of beggars, and being desirous to distribute among them all the money he had about him, finds that if he gave sixpence to each he would have 2s. too little; but that by giving each a groat, he would have 2s. 8d. over; how many beggars were there, and what sum had the gentleman in his pocket?

There were 28 beggars, and the gentleman had in his pocket 12 shillings; for if 28 be multiplied by 6, the product will be 168, from which if 2*s*. or 24 pence, be subtracted, as he wanted 24 pence to be able to give each sixpence, the remainder will be 144 pence = 12 shillings; but by giving each of the beggars 4 pence, he had occasion only for 112 pence, or 4 times 28; consequently he had 32 pence, or 2*s*. 8*d*. remaining.

PROBLEM III.

A gentleman purchased for 110*l. a lot of wine, consisting of* 100 *bottles of Burgundy, and* 80 *of Champagne; and another purchased at the same price, for the sum of* 95*l.* 85 *bottles of the former, and* 70 *of the latter: what was the price of each kind of wine?*

It will be found that the Burgundy cost 10*s*. per bottle, and the Champagne 15*s*. as may be easily proved.

PROBLEM IV.

A gentleman, on his death-bed, gave orders in his will, that if his lady, who was then pregnant, brought forth a son, he should inherit two thirds of his property, and the widow the other third; but that if she brought forth a daughter, the mother should inherit two thirds, and the daughter one third; the lady however was delivered of two children, a boy and a girl, what was the portion of each?

The only difficulty in this problem, is to discover in what manner the testator would have disposed of his property, had he foreseen that his lady would have been delivered of two children. It has generally been explained in the following manner: As the testator desired that if his wife brought forth a boy, the latter should have two thirds of his property, and the mother the other third, it hence follows that his intention was to give the son a portion double to that of the mother; and as he gave orders that in case she brought forth a daughter, the mother was to have two thirds of the property, and the daughter the other, there is reason to conclude that he intended the portion of the mother to be double that of the daughter. Consequently, to combine these two conditions, the property must be divided in such a manner, that the son may have twice as much as the mother, and the mother twice as much as the daughter. If we therefore suppose the property to be 30000*l.* the share of the son will be $17142\frac{6}{7}$*l.* that of the mother $8571\frac{3}{7}$*l.* and that of the daughter $4285\frac{5}{7}$*l.*

As a supplement to this problem, another is differently proposed. In case the mother should be delivered of two sons and a daughter, in what manner must the property be divided?

In our opinion, no other answer can be given to this question, than what would be given by the gentlemen of the bar; that the will, in such a case, would be void; for a child having been omitted in the will, all the laws with which we are acquainted would pronounce its nullity: 1st Because the law is precise, and 2d Because it is impossible to determine what would have been the disposition of the testator, had he had two sons, or had he foreseen that his wife would be delivered of two.

PROBLEM V.

A brazen lion, placed in the middle of a reservoir, throws out water from its mouth, its eyes and its right foot. When the water flows from its mouth alone, it fills the reservoir in 6 hours; from the right eye it fills it in 2 *days; from the left eye in 3, and from the foot in 4. In what time will the bason be filled by the water flowing from all these apertures at once?*

To solve this problem it must be observed, that as the lion, when it throws the water from its mouth, fills the bason in 6 hours, it can fill $\frac{1}{6}$ of it in an hour; and that as it fills it in 2 days when it throws the water from its right eye, it can fill $\frac{1}{48}$ of it in an hour. It will be found, in like manner, that it can fill $\frac{1}{72}$ of it in an hour when the water flows from its left eye, and $\frac{1}{96}$ when it flows from its foot. By throwing the water from all these apertures at once, it furnishes in an hour $\frac{1}{6} + \frac{1}{48} + \frac{1}{72} + \frac{1}{96}$, and these fractions added together are equal to $\frac{61}{288}$. We must therefore make the following proportion: If $\frac{61}{288}$ are filled in one hour or 60 minutes, how many minutes will the whole bason, or $\frac{288}{288}$, require: Or, as 61 is to 288, so is 1 hour to the answer, which will be 4h. 43m. $16\frac{44}{61}$ seconds.

PROBLEM VI.

A mule and an ass travelling together, the ass began to complain that her burthen was too heavy. "Lazy animal," said the mule, "you have little reason to complain; for if I take one of your bags, I shall have twice as many as you, and if I give you one of mine, we shall then have only an equal number."—With how many bags was each loaded?

This problem, which is one of those commonly proposed to beginners in algebra, is taken from a collection of Greek epigrams, known under the name of The Anthology; and has been translated, almost literally, into Latin as follows:

Una cum mulo vinum portabat asella,
Atque suo graviter sub pondere pressa gemebat.
Talibus at dictis mox increpat ipse gementem:
Mater, quid luges, teneræ de more puellæ?

Dupla tuis, si des mensuram, pondera gesto:
At si mensuram accipias, æqualia porto.
Dic mihi mensuras, sapiens geometer, istas?

The analysis of this problem has also been expressed in indifferent Latin verses, which we shall here give, to gratify the reader's curiosity.

Unam asina accipiens, amittens mulam et unam
Si fiant æqui, certe utrique anté duobus
Distabant a se. Accipiat si mulus at unam,
Amittatque asina unam, tunc distantia fiet
Inter eos quatuor. Muli at cum pondera dupla
Sint asinæ, huic simplex, mulo est distantia dupla.
Ergo habet hæc quatuor tantum, mulusque habet octo.
Unam asinæ si addas, si reddat mulus et unam,
Mensuras quinque hæc et septem mulus habebunt.

That is: "As the mule and the ass will both have equal burthens when the former gives one of his measures to the latter, it is evident that the difference between the measures which they carry is equal to 2. Now if the mule receives one from the ass, the difference will be 4; but in that case the mule will have double the number of measures that the ass has; consequently the mule will have 8, and the ass 4. If the mule then gives one to the ass, the latter will have 5 and the former 7:" these were the number of the measures with which each was loaded, and which solve the problem.

This problem, which might be expressed in a great variety of forms, is not the only one furnished by the Greek Anthology. The following are a few more, translated into the Latin verse by Bachet de Meziriac, who inserted them in a note to one of the problems of Diophantus.

1.

Aurea mala ferunt Charites, æqualia cuique
Mala insunt calatho: Musarum his obvia turba

Mala petunt, Charites cunctis æqualia donant;
Tunc æqualia tres contingit habere, novemque.
Dic quantum dederint numerus sit ut omnibus idem?

That is: "The three Graces, carrying each an equal number of oranges, were met by the nine Muses, who asked for some of them; and each Grace having given to each Muse the same number, it was then found that they had all equal shares: How many had the Graces at first?"

The least number which will answer this problem, is 12; for if we suppose that each Grace gave one to each Muse, the latter would each have three; and there would remain 3 to each Grace.

The numbers 24, 36, 48, &c, will also answer the question; and, after the distribution is made, each of the Graces and Muses will have 6, or 9, or 12, &c.

II.

Dic, Heliconiadum decus, ó sublime Sororum
Pythagora! tua quot tyrones tecta frequentent,
Qui, sub te, sophiæ sudant in agone magistro?
Dicam; tuque animo mea dicta, Polycrates hauri:
Dimidia horum pars præclara mathemata discit,
Quarta immortalem naturam nôsse laborat,
Septima, sed tacitè, sedet atque audita revolvit;
Tres sunt fæminei sexûs.

"Tell me, illustrious Pythagoras, how many pupils frequent thy school? One half, replied the philosopher, study mathematics, one fourth natural philosophy, one seventh observe silence, and there are three females besides."

The question here is, to find a number, the $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{7}$ of which + 3, shall be equal to that number. It may be easily replied that this number is 28.

III.

Dic quota nunc hora est? Superest tantùm ecce diei
Quantum bis gemini exactâ de luce trientes.

"A person being asked what o'clock it was, replied, the hours of the day which remain, are equal to $\frac{1}{7}$ of those elapsed."

If we divide the day, as the ancients did, into 12 equal portions, the question will be to divide that number into two such parts, that $\frac{1}{7}$ of the first may be equal to the second: in this case the result will be $5\frac{1}{7}$ for the number of the hours elapsed; and consequently for the remainder of the day $6\frac{6}{7}$ hours.

IV.

Hic Diophantus habet tumulum, qui tempora vitæ
Illius mirâ denotat arte tibi.
Egit sextantem juvenis, lanugine malas
Vestire hinc cœpit parte duodecimâ.
Septante uxori post hæc sociatur et anno
Formosus quinto nascitur inde puer.
Semissem ætatis postquam attigit ille paternæ,
Infelix subitâ morte peremptus obit.
Quatuor æstates genitor lugere superstes
Cogitur, hinc annos illius assequere.

"This is the epitaph of the celebrated mathematician Diophantus. It tells us, that Diophantus passed the sixth part of his life in childhood, and the twelfth part in the state of youth; that after a seventh part of his life and five years more were elapsed, he had a son, who died when he had attained to half the age of his father, and that the latter survived him only four years."

To resolve this problem, we must find a number, the $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{7}$ and $\frac{1}{2}$ of which $+ 5 + 4$, shall be equal to the number itself. This number is 84.

V.

Qui jaculamur aquas tres hic adstamus Amores;
Sed variè liquidas Euripo immittimus undas:
Dexter ego, summis et quæ mihi manat ab alis
Ipsum lympha replet solo sextante diei;

Quatuor ast horis lævus versâ influit urnâ;
Dimidiatque diem medius dum fundit ab arcâ:
Dic, agé, quàm paucis Euripum implebimus horis,
Ex arcâ simul atque alis urnâque fluentes?

"Three Cupids pour water into a bason, but not equal quantities in equal times. One of them can fill it in the sixth of a day, another in four hours, and the third in half a day. How long will they require to fill it when they all pour water into it at the same time?"

This problem is of the same nature as that respecting the brazen lion, of which a solution has been given. If we suppose the day to be divided into 12 hours, it will be found that the three Cupids can fill the bason in $\frac{1}{11}$ of a day, or a little more than an hour.

PROBLEM VII.

The sum of 500l. having been divided among four persons, it was found that the shares of the first two amounted to 285l.; those of the second and third to 220l.; those of the third and fourth to 215l.; and that the share of the first was to that of the last as 4 to 3. What was the share of each?

The solution of this problem is exceedingly easy. The first had 160*l.* the second 125*l.* the third 95*l.* and the fourth 120*l.*

It is to be observed, that without the last mentioned condition, or a fourth one of some kind or other, the problem would be indeterminate; that is to say, would be susceptible of a great many answers: the last condition however limits it to one only.

PROBLEM VIII.

A labourer hired himself to a gentleman on the following conditions: for every day he worked he was to receive 2s. 6d. but for every day he remained idle he was to for-

feit 1*s*. 3*d*.: *after* 40 *days' service he had to receive* 2*l*. 15*s*. *How many days did he work, and how many remain idle?*

He worked 28 days of the 40, and remained idle 12.

PROBLEM IX.

A bill of exchange, of 2000*l*., *was paid with half guineas and crowns; and the number of the pieces of money amounted to* 4700; *how many of each sort were employed?*

There were 3000 half guineas and 1700 crowns.

PROBLEM X.

A gentleman, having lost his purse, could not tell the exact sum it contained, but recollected that when he counted the pieces two by two, or three by three, or five by five, there always remained one; and that when he counted them seven by seven, there remained nothing. What was the number of pieces in his purse?

It may be readily seen that, to solve this problem, nothing is necessary but to find a number which when divided by 7 shall leave no remainder; and which when divided by 2, 3, or 5, shall always leave 1. Several methods may be employed for this purpose; but the simplest is as follows:

Since nothing remains when the pieces are counted seven by seven, the number of them is evidently some multiple of 7; and since 1 remains when they are counted two by two, the number must be an odd multiple: it must therefore be some of the series 7, 21, 35, 49, 63, 77, 91, 105, &c.

This number also when divided by 3 must leave unity; but in the above series 7, 49 and 91, which increase arithmetically, their difference being 42, are the only numbers that have the above property. It appears likewise, that if 91 be divided by 5, there will remain 1; and we may thence conclude that the first number which answers the question is 91; for it is a multiple of 7, and being divided by 2, 3 or 5, the remainder is always 1.

Several more numbers, which answer this question, may be found by the following means: continue the above progression, in this manner: 7, 49, 91, 133, 175, 217, 259, 301, until you find another term divisible by 5, that leaves unity; this term will be 301, and will also answer the conditions of the problem; but the difference between it and 91 is 210, from which it may be concluded, that if we form the progression, 91, 301, 511, 721, 931, 1141, &c. all these numbers will answer the conditions of the problem also.

It would therefore be still uncertain what money was in the purse, unless the owner could tell nearly the sum it contained. Thus, for example, if he should say that there were about 500 pieces in it, we might easily tell him that the number was 511.

Let us now suppose that the owner had said, that when he counted the pieces two by two there remained 1; that when he counted them three by three there remained 2; four by four, 3; five by five, 4; six by six, 5; and, in the last place, that when he counted them seven by seven, nothing remained.

It is here evident that the number, as before, must be an odd multiple of 7, and consequently one of the series 7, 21, 35, 49, 63, 77, 91, 105, &c. But the numbers 35 and 77, of this series, answer the condition of leaving 2 as a remainder when divided by 3, and their difference is 42. For this reason we must form a new arithmetical progression, the difference of which is 42, viz. 35, 77, 119, 161, 203, 245, 287, &c.

We must then seek for two numbers in it, which when divided by 4, shall leave 3 as remainder. Of this kind are the numbers 35, 119, 203, 287; and therefore we must form a new progression, the difference of the terms of which is 84, viz, 35, 119, 203, 287, 371, 455, 539, 623, &c. In this new progression we must seek for two terms, which when divided by 5, shall leave 4; and it will be readily seen that these numbers are 119 and 539, the difference of which is 420. A series of terms therefore which answer all the conditions of the problem except 1, is 119, 539, 959, 1379, 1799, 2219, 2639, &c. But the last condition of the problem is, that the required number when divided by 6, leaves 5 as remainder. This property belongs to 119, 959, 1799, &c, always adding 840; consequently the number sought, is one of those in that progression. For this reason, as soon as we know nearly within what limits it is contained, we shall be able to determine it.

If the owner therefore of the purse had said, that it contained about 100 pieces, the number required would be 119; if he had said there were nearly 1000, it would be 959, &c.

REMARK.

The solution of this problem, according to the method taught by Ozanam, would be imperfect. For after finding the smallest number which answers the conditions of the problem, viz, 119, he would merely say, that to obtain the other numbers which answer them, the numbers 2, 3, 4, 5, 6, 7, ought to be successively multiplied together, and their product 5040 added to 119, the first number found: this would give the number 5159, which would answer the proposed conditions also. But it may be readily seen, that there are several other numbers, between 119 and 5159, which answer these conditions, viz, 959, 1799, 2639, 3479, 4319.

In treating of chronology, we shall give the solution of another problem of the same kind: viz, To find any year of the Julian period, the golden number, cycle of the sun, and cycle of indiction, for that year, being given.

PROBLEM XI.

A sum of money, placed out at a certain interest, increased in 8 months, to 3616l. 13s. 4d. And in two years and a half it amounted to 3937l. 10s. What was the original capital, and at what rate of interest was it placed out?

That young algebraists may have an opportunity of exercising their own ingenuity, we shall here give the answer only of this problem. By employing the proper means of analysis, they will find that the capital was 3500*l.*, and the rate of interest 5 per cent.

PROBLEM XII.

Three women went to market to sell eggs; the first of whom sold 10, *the second* 25, *and the third* 30, *all at the same price. As they were returning they began to reckon how much money they carried back, and it was found that each had the same sum: how many eggs did they sell, and at what price?*

It is evident that to make what is announced in this problem possible, these women must have sold their eggs at two different times, and at different prices; for if the one who had the least number of eggs sold a very small number at the lowest price, and the remainder at the highest, while the one who had the greatest number sold the greater part at the lowest price, and could sell only a very small number at the highest, it may be easily seen that they might have got equal sums of money.

The question then is, to divide each of the numbers 10, 25, 30, into two such parts, that if the first part of each be multiplied by the first price, and the second by the second, the sum of the two products shall be equal.

This problem is indeterminate, and is susceptible of ten different solutions. It is, in the first place, necessary that the difference of the prices of the first and the second sale shall be an exact divisor of 15, 20, 5, the differences of the three numbers given; but the least divisor of these three numbers is 5, and for this reason the prices must be 6 and 1, or 7 and 2, or 8 and 3, &c.

If we suppose the two prices to be 6 and 1, we shall have seven different solutions, as seen in the following table:

	Women	1st sale	2d sale	total amount
I	1st	4 eggs at 6d.	6 at 1d.	30
	2d	1	24	30
	3d	0	30	30
II	1st	5	5	35
	2d	2	23	35
	3d	1	29	35
III	1st	6	4	40
	2d	3	22	40
	3d	2	28	40
IV	1st	7	3	45
	2d	4	21	45
	3d	3	27	45
V	1st	8	2	50
	2d	5	20	50
	3d	4	26	50
VI	1st	9	1	55
	2d	6	19	55
	3d	5	25	55
VII	1st	10	0	60
	2d	7	18	60
	3d	6	24	60

If we suppose the two prices to be 7 and 2, we shall have also the three following solutions:

	Women	1st sale	2d sale	total amount
I	1st	8 eggs at 7d.	2 at 2d.	60
	2d	2	23	60
	3d	0	30	60
II	1st	9	1	65
	2d	3	22	65
	3d	1	29	65
III	1st	10	0	70
	2d	4	21	70
	3d	2	28	70

It would be needless to try 8 and 3, or any other number, as no solution could be obtained from them, for reasons which will be seen hereafter.

REMARKS.

We are told by M. de Lagny, in the second part of his *Arithmetique Universelle*, p. 456, that this question is susceptible of no more than six solutions; but the author is mistaken, for we have here pointed out ten. As it may afford pleasure to those who are studying algebra, to be made acquainted with the method employed for obtaining them, we think it our duty here to give it.

We shall call the price at which the women sold the first time u; and that at which they sold the second time p.

If x then be the number of the eggs sold by the first woman, at the price u, the number of those sold at the price p, will be $10-x$; the money arising from the first sale will be $x u$, that of the second will be $10 p-p x$, and the sum total will be $x u + 10 p-p x$. If z be the number of eggs sold by the second woman, at the first sale; we shall have $u z$ for the money arising from the first sale, and $25 p-p z$ for that arising from the second, making together $z u + 25 p - p z$.

In like manner, if y represent the number of eggs sold, the first time, by the third woman, we shall have $u y$ for the money arising from the first sale, $30 p-p y$ for that of the second, and for the total of the two sales $u y + 30 p-p y$. But, by the supposition, these three sums are equal; consequently $x u + 10 p-p x = z u + 25 p-p z = u y + 30 p - p y$, from which we obtain the three following new equations:

$$x u-p x = z u-p z + 15 p$$
$$x u-p x = u y-p y + 20 p$$
$$z u-p z = u y-p y + 5 p$$

And, dividing the whole by $u-p$, we have these three others, viz,

$$s = z + \frac{15p}{u-p}$$

$$x = y + \frac{20p}{u-p}$$

$$z = y + \frac{5p}{u-p}$$

from which it may be concluded, that $u-p$ must be a divisor of 15, 20, and 5, otherwise, $\frac{15p}{u-p}$, $\frac{20p}{u-p}$, $\frac{5p}{u-p}$, would not be integral numbers, which it is necessary they should be. But the only number which is a divisor of 15, 20, and 5, is 5, which shows that the prices of the two sales could be only 5 and 0, 6 and 1, 7 and 2, 5 and 3, &c.

It may be easily seen, that the supposition of 5 and 0 will not answer the conditions, since in that case there would have been only one sale.

We must therefore try the second supposition, 6 and 1, viz, $u = 6$ and $p = 1$, which gives for the two last equations, $x = y + 4$, $z = y + 1$.

But we have here three unknown quantities, and only two equations; for which reason one of these unknown quantities must be assumed at pleasure. Let us take y, and first suppose it $= 0$.

This will give $x = 4$, and $z = 1$, and we shall have the first solution, which shows that the first woman sold the first time 4 eggs at 6 pence each, and consequently, the second time 6 at 1 penny each; while the second sold 1 the first time at 6 pence, and the other 24 at 1 penny each, and the third sold all her eggs at the second price: they would then all have 30 pence each.

By making $y = 1$, we shall have the second solution.
By making $y = 2$, we shall have the third.
By making $y = 3$, we shall have the fourth.
By making $y = 4$, we shall have the fifth.
By making $y = 5$, we shall have the sixth.
By making $y = 6$, we shall have the seventh.

We cannot suppose y to be greater than 6, because then we should have $x = 10$; which is impossible, as the first woman has only 10 eggs to sell.

We must therefore proceed to the following supposition, viz, $u = 7$, and $p = 2$, which gives two equations, $x = y + 8$; $z = y + 2$.

If y here be first made $= 0$, we shall have $x = 8$, and $z = 2$, which gives the eighth solution.

By making $y = 1$, we shall have the ninth.

By making $y = 2$, we shall have the tenth.

But y cannot be made greater, for then x would be greater than 10, which is impossible.

It would be useless also to try for the values of u and p, 8 and 3; for these would necessarily give to x a value greater than 10, which cannot be the case.

We may therefore rest assured, that the problem is susceptible of no more solutions than the ten above mentioned.

PROBLEM XIII.

To find the number and the ratio of the weights with which any number of pounds, from unity to a given number, can be weighed in the simplest manner.

Though this problem on the first view seems to belong to mechanics, it may be readily seen that it is only an arithmetical question: for, to solve it, nothing is necessary but to find a series of numbers beginning with unity, which added to or subtracted from each other every way possible, shall form all the numbers from unity to the greatest proposed.

It may be solved two ways; either by addition alone, or by addition combined with subtraction. In the first case, the series of weights which answers the problem, is that of the numbers increasing in double progression; in the second, it is that of those in the triple progression.

Thus, for example, with weights of 1 pound, 2 pounds, 4 pounds, 8 pounds, and 16 pounds, we can weigh any number of pounds up to 31: for, with 2 and 1 we can form 3 pounds; with 4 and 1, 5 pounds; with 4 and 2, 6 pounds; with 4, 2 and 1, 7 pounds, &c. With the addition of a weight of 32 pounds, we can weigh as far as 63 pounds; and so on, doubling the last weight, and deducting from that double unity.

But by employing weights in the triple progression, 1, 3, 9, 27, 81, all weights from 1 pound to 121 can be weighed with them: for, with the second less the first, that is to say putting the first into one scale and the second into the other, we can make two pounds; by putting both in the same scale, we can form 4 pounds; by putting 9 on the one side and 3 and 1 on the other, 5 pounds; by 9 on the one side and 3 on the other, 6 pounds; by 9 and 1 on the one side and 3 on the other, 7 pounds; and so on.

It is here evident that this last method is the simplest, being that which requires the least number of different weights.

Both these progressions are more advantageous than any of the arithmetical ones, as will appear on trial; for if the increasing arithmetical weights, 1, 2, 3, 4, &c, were employed, 15 would be necessary to weigh 120 pounds; to weigh 121 with weights in the increasing progression 1, 3, 5, 7, &c, 11 would be required. No other progression would make up all the weights possible, from 1 pound to the greatest resulting from the whole of the weights. The triple proportion therefore is the most convenient of all.

The solution of this problem may be of great utility in commerce, and the ordinary concerns of life, as it affords the means of weighing any weight whatever with the least possible number of different weights.

PROBLEM XIV.

A country woman carrying eggs to a garrison, where she had three guards to pass, sold at the first half the number she had and half an egg more; at the second, the half of *what remained and half an egg more; and at the third, the half of the remainder and half an egg more: when she arrived at the market place, she had three dozen still to sell: how was this possible, without breaking any of the eggs?*

It would appear, on the first view, that this problem is impossible; for how can half an egg be sold without breaking any? The possibility of it however will be evident when it is considered, that by taking the greater half of an odd number, we take the exact half + $\frac{1}{2}$. It will be found therefore that the woman, before she passed the last guard, had 73 eggs remaining, for by selling 37 of them at that guard, which is the half + $\frac{1}{2}$, she would have 36 remaining. In like manner, before she came to the second guard she had 147; and before she came to the first, 295.

This problem might be proposed also in a different manner as follows:

PROBLEM XV.

A gentleman went out with a certain number of guineas, in order to purchase necessaries at different shops. At the first he expended half his guineas and half a guinea more; at the second, half the remainder and half a guinea more; and so at the third: When he returned he found that he had laid out all his money, without having received any change. How was this possible?

He had 7 guineas, and at the first shop expended 4, at the second 2, and at the third 1; for 4 is the half of seven and $\frac{1}{2}$ more; the remainder being 3, its half is $1\frac{1}{2}$, and $\frac{1}{2}$ more makes 2; but 2 taken from 3 leaves 1, the half of which is $\frac{1}{2}$, and $\frac{1}{2}$ makes 1; consequently nothing more remains.

REMARK.

If the number of places at which the gentleman expended all his money were greater, nothing would be necessary but to raise 2 to such a power, that the exponent should be equal to the number of places, and to diminish it by unity. Thus, if there were 4, as the fourth power of 2 is 16, the required number would be 15; if there were 5, the fifth power of 2 being 32, the required number would be 31.

PROBLEM XVI.

Three persons have each such a number of crowns, that if the first gives to the other two as many as they each have; and if the second and third do the same; they will then all have an equal number, namely 8. How many has each?

The first has 13, the second 7, and the third 4; as may be easily proved, by distributing the crowns of each as announced in the problem.

PROBLEM XVII.

A wine merchant who has only two sorts of wine, one of which he sells at 10s. and the other at 5s. per bottle, being asked for some at 8s. per bottle, wishes to know how many bottles of each kind he must mix together, to form wine worth 8s. per bottle?

The difference between the highest price, 10s. and the mean price required, is 2; and that between the mean price and the lowest is 3; which shows that he must take 3 bottles of the wine at the highest price, and 2 of that at the lowest. This mixture will form 5 bottles, worth 8s. each.

In problems of this kind, in general, as the difference between the highest price and the mean price, is to the difference between the mean price and the lowest, so is the number of measures at the lowest price, to that of those at the highest which must be mixed together, to have a similar measure at the mean price.

PROBLEM XVIII.

A gentleman is desirous of sinking 100,000l. which, together *with the Interest, is to become extinct at the end of 20 years, on condition of receiving a certain annuity during that time. What sum must the gentleman receive annually, supposing interest to be at the rate of five per cent?*

The sum which the gentleman ought to receive annually, is 8014*l.* 19*s.* 2*d.* 1·7 *farth.*

If the number of years were small, for example 5, this problem might be resolved, without algebra, by the retrograde method, and false position; for if we suppose the sum, which at the last year exhausts the capital and interest, to be 10,000*l.* we shall find that the capital alone at the commencement of that year was $9523\frac{17}{21}$*l.*; and if we add the 10,000*l.* which were paid at the end of the year preceding the last, the sum $19523\frac{17}{21}$*l.* will be the capital increased with the interest of the fourth year; consequently, the capital at the beginning of that year was only $18594\frac{46}{441}$*l.*; whence it follows, that before the payment, at the end of the third year, the sum was $28594\frac{46}{441}$*l.*, which represented the capital increased with the interest of the third year. By thus going back to the commencement of the first year, the original capital will be found to be 43294*l.* 15*s.* 4*d.* We must then make the following proportion: As this capital, is to the supposed sum of 10,000*l.* so is the sum to be sunk, on the above conditions, to the annuity, or sum to be received every year.

But it may be readily perceived, that, in the case of 20 or 30 years, this method would require very long calculations, which are greatly shortened by algebra*.

* If a be the capital, m the interest, and n the number of years; the annuity or sum to be received every year, will be $\frac{a \times (1+\frac{1}{m})^n}{m \times (1+\frac{1}{m})^n - m}$, which in the case of 20 years, and allowing interest to be at 5 *per cent.* (m being then $= 20$) will be found $= a \times \frac{2\cdot6584}{33\cdot1680} = \frac{a}{12\cdot4764}$.

PROBLEM XIX.

What is the interest with which any capital whatever would be increased, at the end of a year, if the interest due at every instant of the year were itself to become capital and to bear interest?

This problem, to be well understood, has need of explanation. A person might place out his money under this condition, that the interest due at the end of a month, which at the interest of 5 per cent would make a 60th of the capital, should be added to the capital, and bear interest the following month at the same rate; that at the expiration of this month, the interest of the above sum, which would be a 60th + $\frac{1}{3600}$ of the original capital, should be still added to the capital, increased by the interest of the first month, and bear interest the following month, and so on to the end of the year.

What is done here in regard to a month, might be done in regard to a day, an hour, a minute, or even a second, which may be considered as a part of the day infinitely small: the question then is to know, what in this case would be the interest produced by the capital at the end of the year, the interest of the first second being at the rate of five per cent, or $\frac{1}{20}$th.

It might be supposed, on the first view, that this compound and super-compound interest would greatly increase the 5 per cent, and yet it will be found that it produces an increase scarcely sensible; for if the capital be 1, the same capital increased with simple interest, at five per cent, will be $1 + \frac{1}{20}$, or $1 + \frac{50}{1000}$, and when increased with the interest accumulated every second, it will be $1\frac{52}{1000}$, or rather when expressed more exactly, $1\frac{05127}{1000000}$.

PROBLEM XX.

A dishonest butler, every time he went into his master's cel-

lar, stole a pint from a particular cask, which contained 100 pints, and supplied its place by an equal quantity of water. At the end of 30 days, the theft being discovered, the butler was discharged. Of what quantity of wine did he rob his master, and how much remained in the cask?

It may be readily seen that the quantity of wine which the butler stole did not amount to 30 pints; for the second time that he drew a pint from the cask, taking the hundredth part of what it contained, it had already in it a pint of water, and as he each day substituted for the liquor he stole a pint of water, he every day took less than a pint of wine. To resolve, therefore, the problem, nothing is necessary but to determine in what progression the wine which he every day stole decreased.

For this purpose, we must first observe, that after the first pint of wine was drawn, there remained in the cask no more than 99 pints, and the pint of water which had been added. When a pint therefore was drawn from the mixture, it was only $\frac{99}{100}$ of a pint of wine; but before the pint was drawn, the cask contained 99 pints of wine; consequently, after it was drawn, there remained 99 pints $-$ $\frac{99}{100}$, that is to say, $\frac{9801}{100}$ or 98 pints $+ \frac{1}{100}$. When the third pint was drawn, the wine contained in it would be only $\frac{98}{100} + \frac{1}{10000}$, which being taken from the quantity of wine in the cask, viz. 98 $\frac{1}{100}$ pints, would leave $\frac{970299}{10000}$ or 97 pints $+ \frac{299}{10000}$.

It must here be remarked, that $\frac{9801}{100}$ is the square of 99 divided by 100; and that $\frac{970299}{10000}$ is the cube of 99 divided by the square of 100, and so on; consequently, when the second pint is drawn, the wine remaining will be the square of 99 divided by the first power of 100; after the third, it will be the cube of 99 divided by the square of 100, &c. Whence it follows, that after the 30th pint is drawn, the quantity of wine remaining will be the 30th power of 99 divided by the 29th power of 100. But it may be found, by logarithms, that this quantity is $73\frac{97}{100}$, consequently the quantity of wine stolen is 26.3 *.

PROBLEM XXI.

A and B can perform a certain piece of work in 8 days, A and C can do it in 9 days, and B and C in 10 days: how many days will each of them require to perform the same work, when they labour separately?

A will perform it in $14\frac{34}{49}$ days; B in $17\frac{23}{41}$ days, and C in $23\frac{7}{31}$ days.

PROBLEM XXII.

An Englishman owes a Frenchman 1l. 11s.; but has no other money to pay his debt than seven-shillings pieces, and the Frenchman has only French crowns, valued at five shillings. How many seven-shillings pieces must the Englishman give to the Frenchman, and how many crowns must the latter give to the former, that the difference shall be equal to 31 shillings, in favour of the Frenchman, so that the debt may be paid?

The simplest numbers that answer this question, are 8 seven-shillings pieces, and 5 crowns; for 8 seven-shillings pieces make 56 shillings, and 5 crowns make 25; consequently their difference, of which the Frenchman has the advantage in this kind of exchange, is 31 shillings.

This problem is susceptible of an infinite number of solutions; for it will be found that the same result may be obtained with 13 seven-shillings pieces and 12 crowns; 18 seven-shillings pieces and 19 crowns; always increasing

* If the usual method of calculation were employed, it would be necessary to find the 30th power of 99, which would contain not less than 59 figures; and to divide it by unity followed by 58 ciphers; whereas, if logarithms be used, nothing is necessary but to multiply the logarithm of 99 by 30, which will give 598690560, and to subtract the product of the logarithm of 100 multiplied by 29, which is 580000000. The remainder 18690560, is the logarithm of the required quantity; which, in the tables, will be found to be nearly $73\frac{97}{100}$.

the number of the seven-shillings pieces by 5, and that of the crowns by 7.

REMARK.

For the sake of young algebraists, we shall here give the analytical solution of this problem. Let x represent the number of the seven-shillings pieces, and y that of the crowns; $7x$ then will be the sum given by the Englishman, and that given by the Frenchman will be $= 5y$. But as the difference of these two sums must be equal to 31, we shall have $7x - 5y = 31$ shillings; consequently $7x = 31 + 5y$, and $x = \frac{31 + 5y}{7} = 4 + \frac{3 + 5y}{7}$ shillings. But x is a whole number, and 4 being one also, $\frac{3 + 5y}{7}$ must likewise be a whole number. We shall suppose it equal to u; then $7u = 3 + 5y$, and $y = \frac{7u - 3}{5} = u + \frac{2u - 3}{5}$. But y and u, by the supposition, are whole numbers, $\frac{2u - 3}{5}$ must be one also: u then must be of such a nature, that when multiplied by 2, and the product diminished by 3, the remainder shall be divisible by 5. But the first number which has this property is 4; for its product by 2 is 8, and 8 diminished by 3 is 5, which is divisible by 5; and the quotient, 1 added to u or 4, gives 5 for the value of y. It will now be easy to discover the value of x, by observing that $x = 4 + \frac{3 + 5y}{7}$; for if we here substitute for y the value of it already found, that is to say 5, we shall have 8 for the value of x.

The second value of u, which answers the required conditions, is 9; for 2 times 9 is 18, and 18 less 3 is $= 15$, which divided by 5 gives 3; the second value therefore of y is 12, and the corresponding value of x will be found to be 13.

The third value of u, which resolves the question, is 14, which gives for the corresponding values of y and x, 19 and 18. The numbers of crowns therefore, which resolve the question ad infinitum, are 5, 12, 19, 26, 33, 40, &c. always increasing by 7. And the corresponding numbers of seven-shillings pieces are 8, 13, 18, 23, 28, 33, &c. always increasing by 5.

CHAPTER XII.

Of Magic Squares.

THE name magic square is given to a square divided into several other small equal squares or cells, filled up with the terms of any progression of numbers, but generally an arithmetical one, in such a manner, that those in each band, whether horizontal, or vertical, or diagonal, shall always form the same sum.

There are also squares in which the product of all the terms in each horizontal, or vertical, or diagonal band, is always the same. We shall speak of these also, but in a cursory manner, because they are attended with as little difficulty as the former.

These squares have been called *magic squares*, because the ancients ascribed to them great virtues; and because this disposition of numbers formed the bases and principle of many of their talismans.

According to this idea, a square of one cell, filled up with unity, was the symbol of the deity, on account of the unity and immutability of God; for they remarked that this square was by its nature unique and immutable; the product of unity by itself being always unity.

The square of the root 2 was the symbol of imperfect matter, both on account of the four elements, and of the impossibility of arranging this square magically, as will be shewn hereafter.

A square of 9 cells was assigned or consecrated to Saturn; that of 16 to Jupiter; that of 25 to Mars; that of

36 to the Sun; that of 49 to Venus; that of 64 to Mercury; and that of 81, or nine on each side, to the Moon.

Those who can find any relation between the planets and such an arrangement of numbers, must no doubt have minds strongly tinctured with superstition; but such was the tone of the mysterious philosophy of Jamblichus, Porphyry, and their disciples. Modern mathematicians, while they amuse themselves with these arrangements, which require a pretty extensive knowledge of combination, attach to them no more importance than they really deserve.

Magic squares are divided into even and odd. The former are those the roots of which are even numbers, as 2, 4, 6, 8, &c; the latter, of those the roots of which are odd, and which by a necessary consequence have an odd number of cells; such are the squares of 3, 5, 7, 9, &c. As the arrangement of the latter is much easier than that of the former, we shall first treat of them.

§ I.

Of Odd Magic Squares.

There are several rules for the construction of these squares; but, in our opinion, the simplest and most convenient is that which, according to M. de la Loubere, is employed by the Indians of Surat, among whom magic squares seem to be held in as much estimation as they were formerly among the ancient visionaries, before mentioned.

We shall here suppose an odd square, the root of which is 5, and that it is required to fill it up with the first 25 of the natural numbers. In this case, begin by placing unity in the middle cell of the horizontal band at the top; then proceed from left to right, ascending diagonally, and when you go beyond the square, transport the next number 2 to the lowest cell of that vertical band to which it belongs;

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

set 3 in the next cell, ascending diagonally from left to right, and as 4 would go beyond the square, transport it to the most distant cell of the horizontal band to which it belongs; set 5 in the next cell, ascending diagonally from left to right, and as the following cell, where 6 would fall, is already occupied by 1, place 6 immediately below 5; place 7 and 8 in the two next cells, ascending diagonally, as seen in the figure; and then, in consequence of the first rule of transposition, set 9 at the bottom of the last vertical band; then 10, in consequence of the second, in the last cell on the left of the second horizontal band; then 11 below it, according to the third rule: after which continue to fill up the diagonal with the numbers 12, 13, 14, 15; and as you can ascend no farther, place the following number 16 below 15; if you then proceed in the same manner, the remaining cells of the square may be filled up without any difficulty, as seen in the above figure.

The following are the squares of 3 and 7 filled up by the same method; and as these examples will be sufficient to exercise such of our readers as have a taste for amusements of this kind, we shall proceed to a few general remarks on the properties of a square arranged according to this principle.

8	1	6
3	5	7
4	9	2

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

1st. According to this disposition, the most regular of all, the middle number of the progression occupies the centre, as 5 in the square of 9 cells, 13 in that of 25, and

25 in that of 49; but this is not necessary in the arrangement of all magic squares.

2d. In each of the diagonals, the numbers which occupy the cells equally distant from the centre are double that in the centre: thus $30 + 20 = 47 + 3 = 28 + 22 = 24 + 26$, &c, are double the central number 25.

3d. The case is the same with the cells centrally opposite, that is to say, those similarly situated in regard to the centre, but in opposite directions both laterally and perpendicularly: thus 31 and 19 are cells centrally opposite, and the case is the same in regard to 48 and 2, 13 and 37, 14 and 36, 32 and 18. But it happens that, according to this magic arrangement, those cells opposite in this manner are always double the central number, being equal to 50, as may be easily proved.

4th. It may be readily seen, that it is not necessary that the progression to be arranged magically should be that of the natural numbers, 1, 2, 3, 4, &c: any arithmetical progression whatever, 3, 6, 9, 12, &c, or 4, 7, 10, 13, 16, &c, may be arranged in the same manner.

5th. Nor is it necessary that the progression should be continued: it may be disjunct, and the rule is as follows. If the numbers of the progression, arranged according to their natural order in the cells of the square, exhibit in every direction, vertical and horizontal, an arithmetical progression, they are susceptible of being arranged magically in the same square, and by the same process. Let us take, for example, the series of numbers 1, 2, 3, 4, 5; 7, 8, 9, 10, 11; 13, 14, 15, 16, 17; 19, 20, 21, 22, 23; 25, 26, 27, 28, 29: as these, when arranged in the cells of a square, every where exhibit an arithmetical progression, they may be arranged magically: and indeed, according to the above rule, they may be formed into the annexed magic square.

1	2	3	4	5
7	8	9	10	11
13	14	15	16	17
19	20	21	22	23
25	26	27	28	29

In like manner, and for the same reason, the numbers 1, 6, 11, 16, 21; 2, 7, 12, 17, 22; 3, 8, 13, 18, 23; 4, 9, 14, 19, 24; 5, 10, 15, 20, 25, may be arranged magically, by the same process, as in the annexed figure, and give a square of 25. Of the variations of the same square we shall speak hereafter.

20	28	1	9	17
27	5	8	16	19
4	7	15	23	26
11	14	22	25	3
13	21	29	2	10

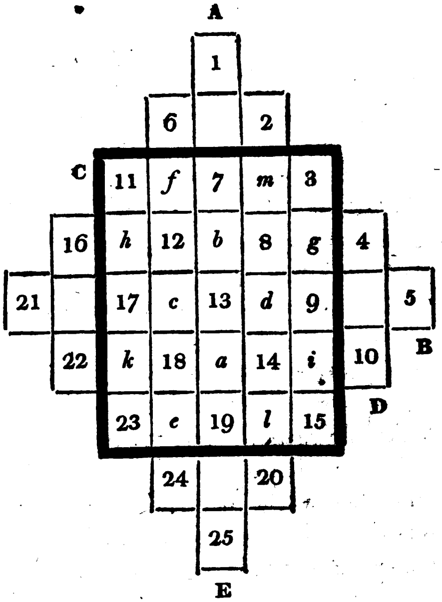
We have likewise the rule of Moscopulus, a modern Greek author, and that of Bachet, who invented one, though unacquainted with any of the above. With these we shall make the reader acquainted also.

9	20	1	12	23
15	21	7	18	4
16	2	13	24	10
22	8	19	5	11
3	14	25	6	17

Moscopulus places unity immediately below the central cell; then sets the following numbers, descending from left to right, and when a number goes without the square, he carries it to the highest cell of the vertical band to which it belongs; he then continues descending obliquely from left to right, and when a number goes beyond the square on the right, he carries it to the most distant cell on the left, from which he continues according to the first rule: if he meets with a cell already filled up, he carries his figure two cells below that last written; and when he arrives at the end of the diagonal, he carries the following number as high as possible in the same vertical band. In the last place, when a number, which ought to be carried two cells lower than the one last placed, goes beyond the square, he carries it to the top of the same band. This description of his method, together with an example, will be sufficient to give a proper idea of it; but it is somewhat more complex than the Indian method.

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

Bachet's rule is as follows: upon each side of the given square raise cells in the form of steps, as seen in the figure; and then, beginning at the highest cell, inscribe all the numbers of the progression, descending diagonally as seen from 1 to 5, and from 6 to 10, &c.



When this is done, transpose into the cell *a*, the next below the centre, the highest number; and in like manner transpose 25 into *b*, the next above the centre; let 5, for the same reason, be transposed to *c*, and 21 to *d*; then transpose 6 to *e*, and 24 to *f*, 20 to *m*, and 2 to *l*, &c. By these means you will obtain the annexed magic square, the sum of each band, whether vertical, or horizontal, or diagonal, will make 65.

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

This rule, though different from that of Moscopulus, gives absolutely the same result.

But all these methods are inferior to the following, invented by M. Poignard, canon of Brussels, and improved and enlarged by M. de la Hire; for the preceding are limited, whereas the one here alluded to is capable of giving an almost infinite number of combinations.

Let it be required, for example, to fill up a square having an odd root, such as 5. Having constructed the square of cells, place in the first horizontal row at the top, the five first numbers of the natural progression, in any order, at pleasure, which we shall here suppose to be 1, 3, 5, 2, 4; then make choice of a number, which is prime to the root 5, and which when diminished by unity does not measure it: let this number be 3; and for that reason begin with the third figure of the series, and count from it to fill up the second horizontal band 5, 2, 4, 1, 3; then begin again by the next third figure, including the 5, that is to say by 4, which will give for the third band 4, 1, 3, 5, 2; by following the same process, we shall then have the series of numbers 3, 5, 2, 4, 1, to fill up the fourth band: continue in this manner, always beginning at the third figure, the preceding included, until the whole square is filled up. This square will be one of the components of the required square, and will be magic; for the sum of each band, whether horizontal, or vertical, or diagonal, is the same as the five numbers of the progression are contained in each without the same figure being ever repeated.

1	3	5	2	4
5	2	4	1	3
4	1	3	5	2
3	5	2	4	1
2	4	1	3	5

Now construct a second geometrical square of 25 cells, in the first band of which inscribe the multiples of the root 5, beginning with a cipher, viz. 0, 5, 10, 15, 20, and in any order at pleasure, such for example as 5, 0, 15, 10, 20: then fill up the square according to the same principle as before, taking care not to assume the same number in the series always to begin with. Thus, for example, as in the former square, the third figure in the series was taken, in the present one the fourth must be assumed; and thus we shall

5	0	15	10	20
10	20	5	0	15
0	15	10	20	5
20	5	0	15	10
15	10	20	5	0

have a square of the multiples, as seen in the annexed figure. This is the second component of the required magic square, and is itself magic, since the sum of each band in every direction is the same.

Now to obtain the magic square required, nothing is necessary but to inscribe in a third square, of 25 cells, the sum of the numbers found in the corresponding cells of the preceding two; for example 5 + 1, or 6, in the first on the left, at the top of the required square: 0 + 3 or 3 in the second, and so on; by these means we shall have the annexed square of 25 cells, which will necessarily be magic.

6	3	20	12	24
15	22	9	1	18
4	16	13	25	7
23	10	2	19	11
17	14	21	8	5

By these means, any of the numbers may be made to fall in any cells at pleasure, for example 1 in the central cell; nothing is necessary for this purpose, but to fill up the middle band with the series of numbers in such a manner, that 1 may be in the centre, as seen in the annexed figure; and then to fill up the rest of the square according to the above principles, beginning at the highest band, when the lowest has been filled up.

2	1	3	4	5
3	4	5	2	1
5	2	1	3	4
1	3	4	5	2
4	5	2	1	3

To form the second square, place a cipher in the centre, as seen in the annexed figure, and fill up the remaining cells in the same manner as before, taking care not to assume the same quantities as in the former, for beginning the bands.

20	5	10	0	15
0	15	20	5	10
5	10	0	15	20
15	20	5	10	0
10	0	15	20	5

In the last place, form a third square by adding together the numbers in the similar cells, and you will have the annexed square, where 1 will necessarily occupy the centre.

22	6	13	4	20
3	19	25	7	11
10	12	1	18	24
16	23	9	15	2
24	5	17	21	8

REMARKS.

I. We must here observe, that when the number of the root is not prime, that is if it be 9, 15, 21, &c, it is impossible to avoid a repetition of some of the numbers, at least in one of the diagonals; but in that case it must be arranged in such a manner, that the number repeated in that diagonal shall be the middle one of the progression; for example 5, if the root of the square be 9; 8 if it be 15; and as the square formed of the multiples will be liable to the same accident, care must be taken, in filling them up, that the opposite diagonal shall contain the mean multiple between 0 and the greatest; for example 36 if the root be 9; 105 if it be 15.

II. The same thing may be done also in squares which have a prime number for their root. By way of example, we shall here form a magic square of the first two of the following ones:

1	2	5	4	3
2	5	4	3	1
5	4	3	1	2
4	3	1	2	5
3	1	2	5	4

10	0	5	15	20
20	10	0	5	15
15	20	10	0	5
5	15	20	10	0
0	5	15	20	10

11	2	10	19	23
22	15	4	8	16
20	24	13	1	7
9	18	21	12	5
3	6	17	25	14

in the first of which 3 is repeated in the diagonal descending from right to left, and in the second 10 is repeated in the diagonal descending from left to right. This however does not prevent the third square, formed by their addition, from being magic.

§ II.

Of Even Magic Squares.

The construction of these squares is attended with more difficulty than that of the odd squares, and the degree of difficulty is different, according as they are evenly even, or oddly even: for this reason we must divide them into two classes.

Squares evenly even, are those the root of which when halved is even, or can be divided by 4 without remainder; of this kind are the squares of 4, 8, 12, &c. The oddly even, are those the root of which when halved gives an odd number; as those of 6, 10, 14, &c.

As the ancients have left us no general rule on this subject, but only some examples of even squares magically arranged, we shall here give the best methods invented by the moderns, and shall begin with squares evenly even.

Let us suppose then, that the annexed square ABCD is to be filled up magically, with the first 16 of the natural numbers: fill up first the two diagonals; and for that purpose begin to count the natural numbers, in order, 1, 2, 3, 4, &c, on the cells of the first horizontal band, from left to right; then proceed to the second band, and when you come to the cells belonging to the diagonals, inscribe the numbers counted as you fall upon them; by which means you will have the arrangement represented in the annexed figure.

A B

1			4
	6	7	
	10	11	
13			16

C D

When the diagonals have been thus filled, to fill up the cells which remain vacant, begin to count the same numbers, proceeding from the angle D in the cells of the lower band, going from right to left, and then in the next above it, and when any cells are found empty, fill them up with the numbers that belong to them: in this manner you will have the square 16 filled up magically, as seen in the annexed figure, and the sum of each band and each diagonal will be 34.

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

REMARKS.

I. The case is here the same as in regard to the odd squares: every progression of numbers which, when arranged in order in a geometrical square, exhibits in every direction, horizontally and vertically, an arithmetical progression, will be susceptible of being arranged magically in the same square.

II. It is not even necessary, in the vertical direction, that the arithmetical proportion should be continued: it may be disjunct; for example, if the numbers 1, 2, 3, 4, 5, 6, 7, 8; 57, 58, 59, 60, 61, 62, 63, 64, when arranged according to their natural order, in a square of 16 cells, exhibit the arithmetical proportions 1, 5, 57, 61; 2, 6, 58, 62, &c; only in the vertical direction, they may be arranged magi-

1	2	3	4
5	6	7	8
57	58	59	60
61	62	63	64

cally in the same square, as seen in the following figure, where the sum of each diagonal band is 130.

We shall now proceed to squares evenly even.

1	63	62	4
60	6	7	57
8	58	59	5
61	3	2	64

Rule for Squares Evenly Even.

We shall suppose that a square of 8 on a side, or 64 cells, is to be filled up, with the first 64 numbers of the natural progression.

First write down these 64 numbers as seen in the two lower lines of the four following periods:

	1	2	3	4	4	3	2	1
I	1	2	3	4	5	6	7	8
	64	63	62	61	60	59	58	57
	4	1	2	3	3	2	1	4
II	9	10	11	12	13	14	15	16
	56	55	54	53	52	51	50	49
	3	4	1	2	2	1	4	3
III	17	18	19	20	21	22	23	24
	48	47	46	45	44	43	42	41
	2	3	4	1	1	4	3	2
IV	25	26	27	28	29	30	31	32
	40	39	38	37	36	35	34	33

The preceding series make 32 couples, each of which forms 65.

Then form the arithmetical progression 1, 2, 3, &c, which must be continued as far as the number that expresses the half of the root; in this case it will be 1, 2, 3, 4, after which form the following three, 4, 1, 2, 3; 3, 4, 1, 2; 2, 3, 4, 1; inscribe, in order, each of these series of numbers over the first terms of each of the above periods of numbers, and as these figures will extend only to the first four, and as there is twice that number, they must be written in an inverse order, over the remaining four.

When this preparation has been made, nothing will be necessary but to write down in order all these numbers, in the cells of the square, taking care, 1st. When the couple of numbers have over them an odd number, to write down the upper number: of this kind are the numbers 1, 3, 6, 8, 10; but when the couple have over them an even number, the lower number must be written down. 2d, When you continue by 33, 34, 35, &c, after the first series has been exhausted, the case will be the contrary.

Thus, the numbers to be successively inscribed in the squares, are 1, 63, 3, 61, 60, 6, 58, 8; which will form the first band: the second will be found by continuing 56, 10, 54, 12, 13, 51, &c; and by proceeding in this manner, you will obtain a square of 8 cells on each side, as here annexed.

1	63	3	61	60	6	58	8
56	10	54	12	13	51	15	49
17	47	19	45	44	22	42	24
40	26	38	28	29	35	31	33
32	34	30	36	37	27	39	25
41	23	42	21	20	46	18	48
16	50	14	52	53	11	55	9
57	7	59	5	4	62	2	64

If the spirit of this method has been properly comprehended it must be seen, that in consequence of it the first and last bands are necessarily filled up with the 16 numbers of the first period, and in such a manner, that the cells centrally opposite form always 65. The case is the same with the second band and the last but one, being filled up with the numbers of the second period, and in the same manner. The same may be said of the third and sixth bands, and the fourth and fifth: it thence follows that the diagonals also must be exact.

Another Rule for Squares evenly even.

Having given, according to M. de la Hire, a very general rule for odd squares, which is capable of producing a great number of variations, we think it our duty to do the same in regard to even squares; especially as it will equally serve for evenly even and oddly even magic squares. It is as follows:

Let it be required, for example, to fill up magically a square of 8 cells on each side.

1	6	5	2	7	4	3	8
8	3	4	7	2	5	6	1
1	6	5	2	7	4	3	8
8	3	4	7	2	5	6	1
8	3	4	7	2	5	6	1
1	6	5	2	7	4	3	8
8	3	4	7	2	5	6	1
1	6	5	2	7	4	3	8

For this purpose, arrange, in the first horizontal band in a square of that kind, the first eight numbers of the arithmetical progression, but in such a manner, that those equally distant from the middle shall form the same sum; viz, that of the root augmented by unity, which in this case is 9; the second band must be the inverse of the first; the third must be like the first; the fourth like the second; and so on alternately, till the half of the square is filled up; after which the other half may be formed by merely reversing the first, as may be seen in the above figure. This will be the first primitive square.

To form the second, fill it up according to the same principle with the multiples of the root, beginning with 0, that is to say, 0, 8, 16, 24, 32, 40, 48, 56, taking care that the extremes shall always make 56; but instead of arranging these numbers in a horizontal direction, they must be arranged vertically, as in the following figure.

48	8	48	8	8	48	8	48
16	40	16	40	40	16	40	16
32	24	32	24	24	32	24	32
0	56	0	56	56	0	56	0
56	0	56	0	0	56	0	56
24	32	24	32	32	24	32	24
40	16	40	16	16	40	16	40
8	48	8	48	48	8	48	8

49	14	53	10	15	52	11	56
24	43	20	47	42	21	46	17
33	30	37	26	31	36	27	40
8	59	4	63	58	5	62	1
64	3	60	7	2	61	6	57
25	38	29	34	39	28	35	32
48	19	44	23	12	45	22	41
9	54	13	50	55	12	51	16

When this is done, add together the similar cells of the two squares, and you will have a square of 8 on each side, as in the last figure above.

Without enlarging farther on squares evenly even, we shall give the simplest method thence deduced, for constructing squares oddly even.

Method for Squares oddly even.

We shall take, by way of example, the square of the root 6. To fill it up, inscribe in it the first six numbers of the arithmetical progression, 1, 2, 3, &c, according to the above method; which will give the first primitive square, as in the annexed figure.

5	6	3	4	1	2
2	1	4	3	6	5
5	6	3	4	1	2
5	6	3	4	1	2
2	1	4	3	6	5
5	6	3	4	1	2

The second must be formed by filling up the cells in a vertical direction, according to the same principle, with the multiples of the root, beginning at 0, viz. 0, 6, 12, 18, 24, 30.

24	6	24	24	6	24
0	30	0	0	30	0
12	18	12	12	18	12
18	12	18	18	12	18
30	0	30	30	0	30
6	24	6	6	24	6

The similar cells of the two squares, if then added, will form a third square, which will require only a few corrections to be magic. This third square is as here annexed.

				A			
	29	12	27	28	7	26	
	2	31	4	3	36	5	
	17	24	15	16	19	14	
C	23	18	21	22	13	20	D
	32	1	34	33	6	35	
	11	30	9	10	25	8	
				B			

To render the square magic, leaving the corners fixed, transpose the other numbers of the upper horizontal band, and of the first vertical one, on the left, by reversing all the remainder of the band; writing 7, 28, 27, 12, instead of 12, 27, &c, and in the vertical one, 32, 23, 17, and 2, from the top downwards, instead of 2, 17, &c.

It will be necessary also to exchange the numbers in the two cells of the middle of the second horizontal band at the top, of the lowest of the second vertical band on the left, and of the last on the right. The numbers in the cells A and B must also be exchanged, as well as those in C and D; by which means we shall have the square corrected and magically arranged.

29	7	28	9	12	26
32	31	3	4	36	5
23	18	15	16	19	20
14	24	21	22	13	17
2	1	34	33	6	35
11	25	10	27	30	8

§ III.

Of Magic Squares with Borders.

Modern arithmeticians have added a new difficulty to the subject of magic squares, by proposing not only to arrange magically in a square a progression of numbers, but by requiring that this square, when lessened by a band on each side, or two or three bands, &c, shall still remain magic; or a magic square being given, to add to it a border of one or more bands, in such a manner, that the enlarged square thence resulting shall be still magic.

To give an example of this construction, let it be required to form a magic square of the root 6, and to fill it up with the natural numbers, from 1 to 36. The first even magic square possible being that of 4 on each side, we shall first arrange it magically, filling it up with the mean terms of the progression, to the number 16, and reserving the first and the last 10 for the border. For the interior square therefore we shall take the numbers 11, 12, &c, as far as 26 inclusively, and shall give them any magic disposition whatever: there will then remain the numbers 1, 2, &c, as far as 10, and 27 as far as 36, for the border.

To dispose these numbers in the border, first place the numbers 1, 6, 31, 36, in the four corners, and in such a manner that diagonally they shall make 37. As each band must make 111, it will be necessary to place in the first band four such numbers, that their sum shall be 104; and as their complements to 37 must be found in the lowest, where there is already 67, it will be necessary that they should together make 44: there are several combinations of these numbers, four and four, which can make 104, and their complements 44; but

1	35	34	5	30	6
33	11	25	24	14	4
28	22	16	17	19	9
8	18	20	21	15	29
10	23	13	12	26	27
31	2	3	32	7	36

it is necessary at the same time that four of those remaining should make 79, to fill up the first vertical band, while their complements make 69 to complete the last. This double condition limits the combination to 35, 34, 30, 5, which may be placed in the first band in any order whatever, provided their complements be placed below each of them in the last band; and the four numbers requisite to fill up the first vertical band will be 33, 28, 10, 8, which may be arranged any how at pleasure, provided the complement of each be placed opposite to it in the corresponding cell on the other side.

It is not absolutely necessary, that 1, 6, 31, 36 should be placed in the four corners of the square: if we suppose them to be filled up, in the same order, with 2, 7, 30, 35, it would be then necessary that the four first numbers should make 102, and their complements 46, while the four last make 79, and their complements 69: but it is found that the four first numbers are 36, 31, 27, 8, and the second 34, 32, 9, 4. The first being arranged any how in the four empty cells of the first band, and their complements below, the second must be arranged in the cells of the first vertical band, and their complements each at the extremity of the same horizontal band; by which means we shall have the new square with a border, as seen above.

2	36	31	27	8	7
34	11	25	24	14	3
32	22	16	17	19	5
9	18	20	21	15	28
4	23	13	12	26	33
30	1	6	10	29	35

If it were required to form a bordered square of the root 8; it would be necessary to reserve for the interior square of 36 cells, the 36 mean numbers of the progression; and they might be formed into a bordered square around the magic square of 16 cells; with the 28 remaining numbers, we might then form a border to the square of 36 cells, &c.

Hence it appears, in what manner we might form a magic square, which when successively lessened by one, two, or three bands, shall still remain magic.

§ IV.

Of another kind of Magic Square in Compartments.

Another property, of which most magic squares are susceptible, is, that they are not only magic when entire, but that when divided into those squares into which they can be resolved, these portions of the original square are themselves magic. A square of 8 cells on a side, for example, formed of four squares, each having 4 for its root, being proposed, it is required that not only the square of 64 shall be disposed magically, but each of those of 16, and that the latter even, however arranged, shall still compose a magic square.

What is here required, is easy; and this is even the simplest method of all for constructing squares that are evenly even, as will appear from what follows.

To construct a square of 64, in this manner, take the first 8 numbers of the natural progression, from 1 to 64, and the 8 last, and arrange them magically in a square of 16 cells; do the same thing with the 8 terms which follow, the first 8 and the 8 which precede the last 8, and by these means you will have a second magic square; form a similar square of the 8 following numbers with their corresponding ones, and another with the 16 means: the result will be four squares of 16 cells, the numbers in which will be equal when added together, either in bands or diagonally; for they will every where be 130.

1	63	62	4	9	55	54	12
60	6	7	57	52	14	15	49
8	58	59	5	16	50	51	13
61	3	2	64	53	11	10	50
17	47	46	20	25	39	38	28
44	22	23	41	36	30	31	33
24	42	43	21	32	34	35	29
45	19	18	48	37	27	26	40

It is therefore evident, that if these squares be arranged side by side, in any order whatever, the square resulting from them will be magic, and the sum in every direction will be 260.

To arrange the square of 9, in this manner; divide the progression, from 1 to 81 inclusively, into nine others, as 1, 10, 19 . . . . 73; 2, 11, 20 . . . . 74; 3, 12, 21 . . . . 75; &c; and arrange each of these progressions magically in a square of 9 cells, marking the first I, the second II, &c. But it will be observed that in these different squares, the sums of the bands and those of the diagonals will be themselves in arithmetical progression; viz, in the square I the sum will be 111, in the square II it will be 114, and so on. If these 9 squares be arranged magically, it may be readily seen that the total will still be magical; but the partial squares cannot be transposed as in the preceding one of 64.

The square of 15 may be resolved into 25 squares of 9 cells. If 25 squares therefore of 9 cells be arranged magically, filling them up with the 25 progressions which may be formed in this manner, 1, 26, 51 . . . . 201; 2, 25, 52 . . . . 202; 3, 28, 53 . . . . 203; &c; these squares will have successively, and in order, for the sums of their bands and their diagonals, 303, 306, 309, &c, to the last, which will make 375 in each of its bands and diagonals. By arranging these 25 squares magically in this manner, calling the first I, the second II, the third III, and the last XXV, you will obtain a magic square; and whatever number of variations the square of 25 cells may be susceptible of, the square of 15 will be capable of receiving as many, being at the same time magic, as well as all the squares of which it is composed.

§ V.

Of the Variations of Magic Squares.

The square having 3 for its root is susceptible of no variation: whatever method may be employed, or whatever arrangement may be given to the numbers of the progression from 1 to 9, the same square will always arise, except that it will be inverted, or turned from left to right, which is not a variation. But this is not the case with the square having 4 for its root, or that of 16 cells: this being susceptible of at least 880 variations, which M. Frenicle has given in his Treatise on Magic Squares.

The square of 5 is susceptible of, at least, 57600 different combinations: for according to the process of M. de la Hire, the 5 first numbers may be arranged 120 different ways in the first band of the first primitive square; and as they may be afterwards arranged in the lower bands, beginning again by two different quantities, this will make 240 variations, at least, in the primitive square; which combined with the 240 of the second, form 57600 variations in the square of 5. But there are doubtless a great many more, for a bordered square of 5 cannot be reduced to the method of M. de la Hire; but one bordered square of 5, the corners remaining fixed, as well as the interior square of 3, may experience 36 variations. What a number therefore of other variations must be produced by changing the interior square and the angles!

A bordered square of 6, when once constructed, the corners remaining fixed, and the interior square being composed of the same numbers, may be varied 4055040 different ways; for the interior square may be varied and differently transposed in the centre 7040 ways: each of the horizontal bands at top and at bottom, the extremities remaining fixed, may be varied 24 ways; for there are four pairs of numbers susceptible of changing their place, which may be combined 24 ways; and there are also four pairs in the vertical bands between the corners. The number of the combinations therefore, is the product of 7040 by 576, the square of 24, which gives 4055040 variations. But the corners may be varied, as well as the numbers assumed to form the interior square; and it hence follows, that the whole number of the variations of a square of 6, while it still remains bordered, is equal to several millions of times the former.

The square of 7, by M. de la Hire's method alone, may be varied 406425600 different ways.

These variations, however numerous, ought to excite no surprise; for the number of dispositions, magic or not magic, of 49 numbers, for example, forms one of 62 figures, of which the preceding is, as we may say, but a part infinitely small.

§ VI.

Of Geometrical Magic Squares.

We have already observed, in the beginning of this chapter, that numbers in geometrical progression might be arranged in the cells of a square, and in such a manner, that the product of these numbers, in each band, whether vertical or horizontal, or diagonal, shall always be the same.

To construct a square of this kind, the same principles must be followed as in the construction of other magic squares; and this may be easily demonstrated from the property of logarithms. Without enlarging further therefore on this subject, we shall confine ourselves to giving one example; it is that of the 9 first terms of the double geometric progression, 1, 2, 4, 8, &c, arranged in a square of 3 cells on each side. The product is evidently the same in every direction, viz. 4096.

128	1	32
4	16	64
8	256	2

REMARKS.

The ingenious Dr. Franklin, it seems, carried this curious speculation further than any of his predecessors in the same way. He constructed both a magic square of squares, and a magic circle of circles, the description of which is as follows. The magic square of squares is formed by dividing the great square as in fig. 1, Pl. 4. The great square is divided into 256 little squares, in which all the numbers from 1 to 256, or the square of 16, are placed in 16 columns, which may be taken either horizontally or vertically. Their chief properties are as follow:

1. The sum of the 16 numbers in each column or row, vertical or horizontal, is 2056.

2. Every half column, vertical and horizontal, makes 1028, or just one half of the same sum 2056.

3. Half a diagonal ascending, added to half a diagonal descending, makes also the same sum 2056; taking these half diagonals from the ends of any side of the square to the middle of it; and so reckoning them either upward or downward; or sideways from right to left, or from left to right.

4. The same with all the parallels to the half diagonals, as many as can be drawn in the great square: for any two of them being directed upward and downward, from the place where they begin, to that where they end, their sums still make the same 2056. Also the same holds true downward and upward; as well as if taken sideways to the middle, and back to the same side again. Only one set of these half diagonals and their parallels, is drawn in the same square upward and downward; but another set may be drawn from any of the other three sides.

5. The four corner numbers in the great square, added to the four central numbers in it, make 1028, the half sum of any vertical or horizontal column, which contains 16 numbers: and also equal to half a diagonal or its parallel.

6. If a square hole, equal in breadth to four of the little squares or cells, be cut in a paper, through which any of the 16 little cells in the great square may be seen, and the paper be laid upon the great square; the sum of all the 16 numbers seen through the hole, is always equal to 2056,

the sum of the 16 numbers in any horizontal or vertical column.

The magic circle of circles, fig. 2, Pl. 4, by the same author, is composed of a series of numbers, from 12 to 75 inclusive, divided into 8 concentric circular spaces, and ranged in 8 radii of numbers, with the number 12 in the centre; which number, like the centre, is common to all these circular spaces, and to all the radii.

The numbers are so placed, that 1st, the sum of all those in either of the concentric circular spaces above mentioned, together with the central number 12, amount to 360, the same as the number of degrees in a circle.

2. The numbers in each radius also, together with the central number 12, make just 360.

3. The numbers in half of any of the above circular spaces, taken either above or below the double horizontal line, with half the central number 12, make just 180, or half the degrees in a circle.

4. If any four adjoining numbers be taken, as if in a square, in the radial divisions of these circular spaces, the sum of these, with half the central number, make also the same 180.

5. There are also included four sets of other circular spaces, bounded by circles that are eccentric with regard to the common centre; each of these sets containing five spaces; and the centres of them being at A, B, C, D. For distinction, these circles are drawn with different marks, some dotted, others by short unconnected lines, &c; or still better with inks of divers colours, as blue, red, green, yellow.

These sets of eccentric circular spaces intersect those of the concentric, and each other; and yet, the numbers contained in each of the eccentric spaces, taken all around through any of the 20, which are eccentric, make the same sum as those in the concentric, namely 360, when the central number 12 is added.

Their halves also, taken above or below the double horizontal line, with half the central number, make up 180.

It is observable, that there is not one of the numbers, but what belongs at least to two of the circular spaces; some to three, some to four, some to five: and yet they are all so placed, as never to break the required number 360, in any of the 28 circular spaces within the primitive circle.

CHAPTER XIII.

Political Arithmetic.

SINCE politicians have acquired juster ideas respecting what constitutes the real strength of states, various researches have been made in regard to the number of the inhabitants in different countries in order to ascertain their population. Besides, as almost all governments have been under the necessity of making loans for the most part on annuities, they have naturally been induced to examine, according to what progression mankind die, that the interest of these loans may be proportioned to the probability of the annuities becoming extinct. These calculations have been distinguished by the name of political arithmetic, and as it exhibits several curious facts, whether considered in a political or a philosophical point of view, we have thought it our duty to give it a place here, to amuse and instruct our readers.

§ I.

Of the Proportion between the Males and the Females.

Many people imagine that the number of the females born exceeds that of the males; but it has long since been proved that the contrary is the case. More boys than girls are born every year; and since the year 1631, a small interval excepted, we have a register of births, in regard to sex, and it has never been observed that the number of the females born even equalled that of the males. It is found, by taking a mean or average term in a great number of years, that the number of the males born is to that of the females, as 18 to 17. This proportion is nearly that which prevails throughout all France; but, to whatever reason owing, it seems at Paris to be as 27 to 26.

This kind of phenomenon is observed, not only in England and in France, but in every other country. We may be convinced of the truth of it by inspecting the monthly and other periodical publications, which at the commencement of every year give a table of the births that have taken place in most of the capital cities of Europe: it may there be seen, that the number of the males born always exceeds that of the females; and consequently it may be considered as a general law of nature.

We may here observe a striking instance of the wisdom of providence, which has thus provided for the preservation of the human race. Men, in consequence of the active life for which they are naturally destined by their strength and their courage, are exposed to more dangers than the female sex; war, long sea voyages, employments laborious or prejudicial to health, and dissipation, carry off great numbers of the males; and it thence results, that if the number born of the latter did not exceed that of the females, the males would rapidly decrease, and soon become extinct.

§ II.

Of the Mortality of the Human Race, according to the different Ages.

In this respect, there is apparently a considerable difference between large towns and the country; but this arises from the women in town rarely suckling their own children: and consequently the greater part of their children being put out to nurse in the country, as it is in the period of childhood that the greatest mortality prevails, it becomes most apparent in the country. To make an exact calculation, it ought to be founded on the deaths which happen in the towns, as well as in the country; and this M. Dupré de St. Maur has endeavoured to do, by comparing the registers of three parishes in Paris, and twelve in the country.

According to the observations of this author, in 23994 deaths, 6454 of them were those of children not a year old; and carrying his researches on this subject as far as possible, he concludes, that of 24000 children born, the numbers who attain to different ages are as follow:

Ages	Number
2	17540
3	15162
4	14177
5	13477
6	12968
7	12562
8	12255
9	12015
10	11861
15	11405
20	10909
25	10259
30	9544
35	8770
40	7929
45	7008
50	6197
55	5375
60	4564
65	3450
70	2544
75	1507
80	807
85	291
90	103
91	71
92	63
93	47
94	40
95	33
96	23
97	18
98	16
99	8
100	6 or 7.

Such then is the condition of the human species, that of 24000 children born, scarcely one half attain to the age of 9; and that two thirds are in their grave before the age of 40; about a sixth only remain at the expiration of 62 years; a tenth after 70; a hundredth part after 86; about a thousandth part attain to the age of 96; and six or seven individuals to that of 100.

We must however observe, that the authors who have treated on this subject differ from each other. According to the table of M. de Parcieux, for example, the half of the children born do not die before 31 years are completed; but according to M. Dupré de St. Maur they are cut off before the commencement of the ninth year. This difference arises from the table of M. de Parcieux having been formed from lists of annuitants, who are always select subjects; for a father never thinks of purchasing an annuity on the life of a child who is sickly, or has a bad constitution. The laws of mortality in these cases therefore are different; and if the one is a general and common law, the other is that which public administrators, who grant annuities, ought to consult with great care, that they may not make loans too burthensome.

§ III.

Of the Vitality of the Human Species, according to the different Ages, or Medium of Life.

When a child is born, to what age may a person bet, on equal terms, that it will attain? Or if the child has already attained to a certain age, how many years is it probable it will still live? These are two questions, the solution of which is not only curious, but important.

We shall here give two tables on this subject; one by M. Dupré de St. Maur, and the other by M. Parcieux; and add to them a few general observations.

TIME TO LIVE.

	M. de St. Maur.		M. de Parcieux.	
AGE.	YEARS.	MONTHS.	YEARS.	MONTHS.
0	8			
1	33		41	9
2	38		42	8
3	40		43	6
4	41		44	2
5	41	6	44	5
6	42		44	3
7	42	3	44	
8	41	6	43	9
9	40	10	43	3
10	40	2	42	8
20	33	5	36	3
30	28		30	6
40	22	1	25	6
50	16	7	19	5
60	11	1	14	11
70	6	2	9	2
75	4	6	6	10
80	3	7	5	
85	3		3	4
90	2		2	2
95		5		6
96		4		5
97		3		4
98		2		3
99		1		2
100		½		1

Two observations here occur, in regard to these tables. The first is respecting the difference between them. In that of M. Parcieux, the time assigned to each age to live is more considerable, and the reason has been already mentioned. We have even suppressed the first year in the table of M. Parcieux, because the difference was by far too great, and in our opinion it arose from two causes. 1st. No one ever thinks of purchasing an annuity for a child in its first year, until the goodness of its constitution has been fully ascertained. 2d. It is not at the birth of a child, but in the course of the first year, towards the middle or end, that such a measure is hazarded; for as annuities remain sometimes several months, and even a whole year, to be filled up, people are not under the necessity of sinking money on so young a life, and have time during the course of several months to acquire some certainty respecting the constitution of the subject. In our opinion therefore, the 34 years of vitality, assigned by M. de Parcieux to a child just born, ought to be considered as applicable to a child from 6 to 9 months old, and more; but it is during the first months of the first year that the life of a child is most uncertain, and that the greatest number die.

The second observation, which is common to both tables, is, that vitality, exceedingly weak at the moment of birth, goes on increasing after that period, till it comes to another, at which it is the greatest; for the chance is less than 3 to 1 that a new born child will attain to the end of its first year*, and one may take an even bet that it has only 8 years to live; but when it has attained to the commencement of the second year, one may bet 6 to 1 that it will attain to the third; and it is an even chance that it will live 33 years. In a word, it is seen by the table of M. Dupré de St. Maur, that it is towards the age of 10 years, or between 10 and 15, that life is most secure. At that period, one may take an even bet that the child will live

* According to the principles explained in treating of probabilities, the probability of a child newly born being alive at the end of a year, is to that of its dying before that period, as the number of the children alive at the end of a year, is to the number of those dead; that is to say, as 17540 to 6460; which is somewhat less than the ratio of 3 to 1. In the other cases the calculation is the same. Take the number of those who have died in the course of the year, and divide by it the number of those alive; this will express what may be betted to 1, that the person who has completed that year will complete another.

43 years; and it is 125 to 1 that it will live a year, or 25 to 1 that it will live five years. Beyond that period the probability of living a year longer decreases. At the age of 20, for example, it is somewhat less than 16 to 1, that the person will not die within the five following years. When a person has reached his sixtieth year, it is no more than 3½ to 1 that he will attain to the beginning of his sixty-fifth year.

§ IV.

Of the Number of Men of different Ages in a given Number.

It may be deduced from the preceding observations, that when the inhabitants of a country amount to a million, the number of those of the different ages will be as follows:

Between 0 and	1 year complete	38740
1	5	119460
5	10	99230
10	15	94530
15	20	88674
20	25	82380
25	30	77650
30	35	71665
35	40	64205
40	45	57230
45	50	50605
50	55	43940
55	60	37110
60	65	28690
65	70	21305
70	75	13195
75	80	7065
80	85	2880
85	90	1025
90	95	335
95	100	82
Above 100 years		3 or 4
	Total	1,000000

Thus in a country peopled with a million of inhabitants, there are about 573460 between the age of 15 and 60; and as nearly one half of them are men, this number of inhabitants could, on any emergency, furnish 250 thousand men capable of bearing arms, even if an allowance be made for the sick, the lame, &c, who may be supposed to be among that number.

§ V.

Of the Proportion of the Births and Deaths to the whole Number of the Inhabitants of a Country.—The Consequences thence resulting.

As it would be difficult to number the inhabitants of a country, and much more to repeat the enumeration as often as it might be necessary to ascertain the population, means have been devised for accomplishing the same object, by determining the proportion which the births and deaths bear to the whole number of the inhabitants; for as registers of births and deaths are regularly kept in all the civilized countries of Europe, we may judge, by comparing them, whether the population has increased or decreased; and in the latter case can examine the causes which have produced the diminution.

It is deduced, for example, from Dr. Halley's tables of the state of the population of Breslaw, about the year 1690, that among 34000 inhabitants, there took place, every year on an average, 1238 births; which gives the proportion of the former to the latter as 27½ to 1. In regard to cities, such as Breslaw, where there is no great influx of strangers, we may therefore adopt it as a rule, to multiply the births by 27½, in order to find the number of the inhabitants.

There appeared, some years ago, that is to say in 1766, a very interesting work on this subject, entitled, *Recherches sur la Population des Généralités d'Auvergne, de Lyon, de Rouen, et de quelque Provinces et Villes du Royaume, &c.*

by M. Messance. By an enumeration of the inhabitants of seventeen small towns or villages, in the generality of Auvergne, compared with the average number of births in the same places, the author shows, that the number of the births is to that of the inhabitants, as 1 to $24\frac{1}{2}$, $\frac{1}{40}$ $\frac{1}{80}$: a similar enumeration, in twenty-eight small towns or villages of the generality of Lyons, gave the ratio of 1 to $23\frac{3}{4}$; and by another made in five small towns or villages of the generality of Rouen, it appeared that the ratio was as 1 to $27\frac{1}{2}$ and $\frac{1}{20}$. But as these three generalities comprehend a very mountainous district, such as Auvergne, another which is moderately so, as the generality of Lyons, and a third which consists almost entirely of plains or cultivated hills, as the generality of Rouen, there is reason to conclude, that these three united afford a good representation of the average state of the kingdom; combining therefore the above proportions, which gives that of 1 to $25\frac{1}{2}$, this will give the proportion of births to the number of the inhabitants, for the whole kingdom, without including the great cities; so that for two births in the year we shall have 51 inhabitants.

But as, in towns of any magnitude, there are several classes of citizens who spend their lives in celibacy, and who contribute either nothing or very little to the population, it is evident that this proportion between the births and effective inhabitants must be greater. M. Messance says, he ascertained, by several comparisons, that the ratio nearest the truth, in this case, is 1 to 28, and that this is the proportion which ought to be employed in deducing, from the number of births, the number of the inhabitants of a town of the second order, such as Lyons, Rouen, &c; which agrees pretty well with what Dr. Halley found in regard to the city of Breslaw.

In the last place, for cities of the first class, or the capitals, of states, such as Paris, London, Amsterdam, &c, where a great many strangers, attracted either by pleasure or business, are mixed with the inhabitants, and where great luxury prevails, which increases the number of those who live in voluntary celibacy, it is very probable that the above ratio must be raised, and that it ought to be carried to 30 or 31.

M. Kerseboom, in his book entitled, *Essai de Calcul politique, concernant la quantité des habitans des provinces de Hollande et de Westfriesland, &c*, printed at the Hague, in 1748, has endeavoured to show that, to obtain the number of the inhabitants in Holland, the number of the births ought to be multiplied by 35. If this be the case, there is reason to conclude, that marriages are less fruitful, or less numerous in Holland, than in France; and this difference may be founded on physical causes.

If these calculations be applied to determining the population of great cities, it will be seen that the opinions entertained in general on this subject are erroneous; for it is commonly said that Paris contains a million of inhabitants; but the number of births there, taking one year with another, never exceeds 19500, which multiplied by 30 gives 585000 inhabitants; if we employ as multiplier the number 31, we shall have 604500, and this is certainly the utmost extent of the population of Paris.

§ VI.

Of some other Proportions between the Inhabitants of a Country.

We shall present to the reader a few more short observations in regard to population. The book, which we quoted in the preceding paragraph, shall still serve us as a guide.

By combining together the three generalities above mentioned, it is found,

1st. That the number of the inhabitants of a country, is to that of the families, as 1000 to 222½; so that 2000 inhabitants give in common 445 families, and consequently

4½ heads on an average for each, or 9 persons for two families. In this respect, those of Auvergne are the most numerous; those of the Lyonnois are next, and those of the generality of Rouen are the least numerous. By taking a mean, it is found also, that in 25 families, there is one where there are 6 or more children.

2d. The number of male children born exceeds, as has been said, that of the females; and this excess continues till a certain age; for example, the number of boys of the age of fourteen, or below, is greater than that of the females of the same age, and in the ratio of 30 to 29. The whole number of the females, however, exceeds that of the males, in the ratio of about 18 to 19. We here see the effect of the great consumption of men, occasioned by war, navigation, laborious employments, and debauchery.

3d. It is found that there are three marriages annually among 337 inhabitants; so that 112 inhabitants produce one marriage.

4th. The proportion of married men or widowers, to married women or widows, is nearly as 125 to 140; and the whole number of this class of society, is to the whole of the inhabitants, as 265 to 631, or as 53 to 126.

5th. According to King and Kerseboom, the number of widowers is to that of widows, as 1 to 3 nearly; so that there are three widows for one widower. This at least is deduced from the enumerations made in Holland and in England. But is the case the same in France? It is to be regretted, that the above mentioned author did not make researches on this subject. In our opinion, however, this proportion is pretty near the truth, and it will excite no astonishment when it is considered that the greater part of the men marry late, in comparison of the women.

6th. If the above proportion between widowers and widows be admitted, it thence follows, that among 631 inhabitants there are 118 married couples, 7 or 8 widowers, and 21 or 22 widows: the remainder are composed of children, people in a state of celibacy, servants, or passengers.

7th. It thence results also, that 1870 married couples give annually 357 children; for a city of 10000 inhabitants would contain that number of married couples, and give 357 annual births. Five married couples therefore, of all ages, produce annually one birth.

8th. The number of servants is to the whole number of the inhabitants, as 136 to 1535 nearly; which is somewhat more than the eleventh part, and less than the tenth.

The number of male servants is nearly equal to that of the female, being in the ratio of 67 to 69; but it is very probable that in large cities, where a great deal of luxury prevails, the proportion is different.

9th. The number of ecclesiastics of both sexes, that is to say, secular as well as regular, comprehending the nuns, is to the inhabitants of the above three generalities, as 1 to 112 nearly; this is contrary to the common opinion, which supposes the proportion to be much greater.

10th. By dividing the territory of these three generalities among their inhabitants, it is found, that the square league would contain 864; but the square league contains 6400 acres: each man therefore, on an average, would have $7\frac{4}{10}$ acres, and each family being composed, one with another, of $4\frac{1}{2}$ heads, $33\frac{1}{2}$ acres would fall to the share of each family. But it is to be observed, that the generality of Rouen, considered alone, is much more populous, since it contains 1264 inhabitants for each square league, which gives to each head no more than five acres.

11th. It appears by the same enumerations, that a very sensible increase in the population has taken place since the beginning of the last century. It is indeed found, that the annual number of the births has been augmented; and by comparing the present period with the commencement of the last century, there is reason to conclude, that the number of the inhabitants is now greater than what it was at the beginning of the century, in the ratio of 1456 to 1350; which makes less than a twelfth, and more than a thirteenth, of increase. This is doubtless owing to the great extent to which agriculture and commerce have been carried, and to the cessation of those wars which so long exhausted the interior of France. The wound given to the nation by the revocation of the edict of Nantes seems healed, and even more; but had it not been for that event, France, in all probability, would contain a sixth more of inhabitants than it did at the commencement of the 18th century; for the number who expatriated in consequence of that revocation amounted perhaps to a twelfth part of the whole people.

§ VII.

Some Questions which depend on the preceding Observations.

The following are some of those questions, in the solution of which the preceding observations may be employed: we shall not explain the principles on which each is resolved; but shall merely confine ourselves to referring to them sometimes, that we may leave to the reader the pleasure of exercising his own ingenuity.

1st. *The age of a man being given, that of 30 for example, what probability is there that he will be living at the end of a determinate number of years, such as 15?*

Seek in the table of the second section for the given age of the person, viz. 30 years, and write down the number opposite to it, which is 11405; then take from the same table the number opposite to 45, which is 7008, and form these two numbers into a fraction, having the latter for its numerator, and the former as its denominator; this fraction will express the probability of a person of 30 years of age living 15 years, or attaining to the age of 45.

The demonstration of this rule is obvious to every one who understands the theory of probabilities.

2d. *A young man 20 years of age borrows 1000l. to be* *paid, capital and interest, when he attains to the age of 25; but in case he dies before that period, the debt to become extinct. What sum ought he to engage to pay, on attaining to the age of 25?*

It is evident that if it were certain he would not die before the age of 25, the sum to be then paid would be the capital increased by 5 years interest, which we here suppose to be simple interest: the sum therefore which in that case he ought to engage to pay, on attaining to the age of 25, would be 1250*l.* But this sum must be increased, in proportion to the danger of the debtor dying in the course of these five years, or in the inverse ratio of the probability of his being alive when they are expired. As this probability is expressed by the fraction $\frac{10259}{10909}$, we must multiply the above sum by this fraction inverted, or by $\frac{10909}{10259}$, which will give 1329*l.* 3*s.* 11*d.* that is to say, 79*l.* 3*s.* 11*d.* for the risk of losing the debt, which certainly cannot be considered as usury.

3d. *A state or an individual having occasion to raise money on annuities, what interest ought to be given for the different ages, legal interest being at the rate of 5 per cent.?*

The vulgar, who are accustomed to burthensome loans, entertain no doubt that 10 per cent. is a great deal for any age below 50, and that such a method of borrowing cannot be advantageous to the state. But this is a great mistake; for it will be found by calculation, employing the before mentioned data, according to the table of M. de Parcieux, that 10 per cent. cannot be allowed before the age of 56. According to the same table, no more than $6\frac{1}{5}$ can be given at the age of 20; $6\frac{1}{2}$ at 25; $6\frac{4}{5}$ at 30; $7\frac{2}{5}$ at 40: $8\frac{4}{5}$ at 50; 10 at 56; $11\frac{1}{10}$ at 60: $16\frac{2}{5}$ at 70; $27\frac{2}{5}$ at 80; $39\frac{1}{10}$ at 85.

It is also a very great error to believe, that on account of the number of persons who lay out money in these loans made by governments, they are soon freed from a part of the annuities by the death of a part of the annuitants.

The slowness of the increase of annuities in tontines, is a sufficient proof of the falsity of this idea; besides, the great number of the persons is the very cause why the extinction of the annuities takes place according to the laws of probability above explained. A fortunate circumstance, at the end of some years, may free an individual from an annuity established on the life of a man aged 30; but if this annuity were divided among 300 persons of nearly the same age, it is certain that he would not be freed from this burthen before the expiration of about 65 years; and at the end of 32 or 33 years, one half of the annuitants would still be living. This M. de Parcieux has shown, in the clearest manner, by examining the lists of different tontines.

4th. *Legal Interest being at 5 per cent.; at whate rate of Interest may an annuity be granted on the lives of two persons, whose ages are given, and payable till the death of the last survivor?*

5th. *What interest may be allowed on a capital, sunk for an annuity on the lives of two persons, whose ages are given, and payable only while both the annuitants are living?*

6th. *A certain person, whose age is given, has an annuity, secured on the public funds, of* 1000*l.; but being in want of money, he is desirous to sell it. How much is it worth?*

7th. *A. aged* 20, *and B. aged* 50, *agree to purchase, on their joint lives, an annuity of* 1000*l., to be equally divided between them, during their lives, with a reversion to the survivor. How much ought each of them to contribute towards the purchase money?*

8th. *How much ought each to contribute, supposing it stipulated between them, that B. the eldest, should enjoy the whole till his death?*

9th. *Legal interest being at* 5 *per cent., what is the worth of an annuity of* 100*l., on the lives of three persons, whose ages are given, and payable till the death of the last survivor?*

10th. *An annuity is purchased for the life of a child, of 3 years of age, on this condition, that the annuity at the end of each year is to be added to the purchase money, till the annuity equals the capital sunk. At what age will the annuity be due, legal interest being at 5 per cent?*

Many people imagine that a capital can be deposited in the bank of Venice on this condition, that nothing is received for the first 10 years, but after that period the annuitant receives an annuity equal to the capital. This however is entirely groundless, as has been shown by M. de Parcieux in his *Addition à l'Essai sur les Probabilités de la durée de la Vie humaine*, published in 1760; for it is there shown, by a calculation, the demonstration of which is evident, that if 100*l.*, for example, were sunk on the life of a child 3 years of age, it could not begin to enjoy an annuity of 100*l.* till it had attained to the age of 45 or 46.

The table of M. de Parcieux presents, on this subject, two things very curious. For example, on the above supposition, if the increase of the annuity were not stopped till the end of 54 years, the person ought to receive 205*l.* per annum during the remainder of his life; if it were not stopped till 58 years, he ought to receive till the time of his death 300*l.*; and by stopping it only at 75 years, he would be entitled to 2900*l.* per annum: in the last place, if the arrears due each year were left, on the like conditions, to accumulate till the 94th year, the annuity for the remainder of the person's life ought to be 134069*l.* 19*s.* 2*d.* a sum which must appear prodigious.

But it may seem astonishing that M. de Parcieux should begin his calculations only at the age of 3 years. It is very true that people do not venture capitals in the purchase of annuities on the lives of new born children; but if ever such an establishment existed at Venice, it is evident that it must have been only on the supposition of the money being risked on the life of a child just born, because great mortality takes place during the first year.

For this reason we have examined what would be the result of such a supposition; and we have found that, if the sum of 100*l.* were sunk, on the above conditions, on the life of a child just born, it ought, according to the table of M. Dupré de St. Maur, to procure it an annuity of 10*l.* 15*s.*; that this sum sunk in like manner, at 8 per cent., at the end of the first year, by adding the first annuity, would give at the end of the second year 11*l.* 11*s.* 7*d.* These 11*l.* 11*s.* 7*d.* sunk at $6\frac{3}{10}$ per cent., which is the interest that might be allowed at the commencement of the third year, would at the end of the third, or the commencement of the fourth, amount to 12*l.* 5*s.* 1*d.* and, by a calculation similar to that of M. de Parcieux, it will be found, that the annuity would be increased to 100*l.* at about the age of 36; which is still very far distant from what is commonly believed.

If legal interest be supposed to be 10 per cent., as it was in the 16th century, it will be found, that it would be only about the 26th year that a person could receive an annuity equal to the capital sunk at the time of his birth.

Those who are desirous of farther information on this subject may consult Demoivre's Essay upon Annuities on Lives, and M. de Parcieux's *Essai sur les Probabilités de la durée de la Vie humaine*, and Dr. Price on Reversionary Payments. The other authors who have treated mathematically on these matters are, Dr. Halley, Sir William Petty, Major Graunt, King, Davenant, Simpson, Maseres; and among the Dutch, the celebrated John de Wit, grand pensionary of Holland, M. Kerseboom, Struyk, &c.

Fig. 1.

Fig. 2.

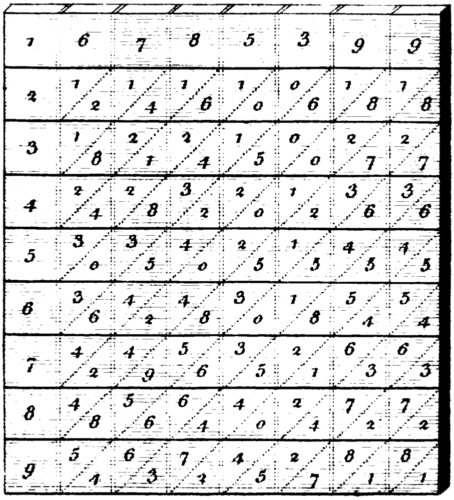


Fig. 3.

Triangular Numbers

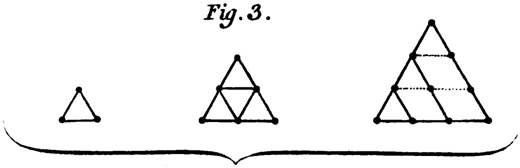
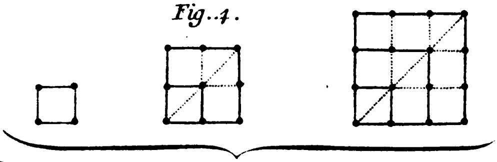


Fig. 4.

Square Numbers



Pentagonal Numbers

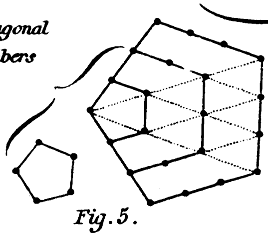


Fig. 5.

Hexagonal Numbers

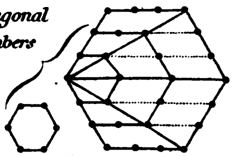
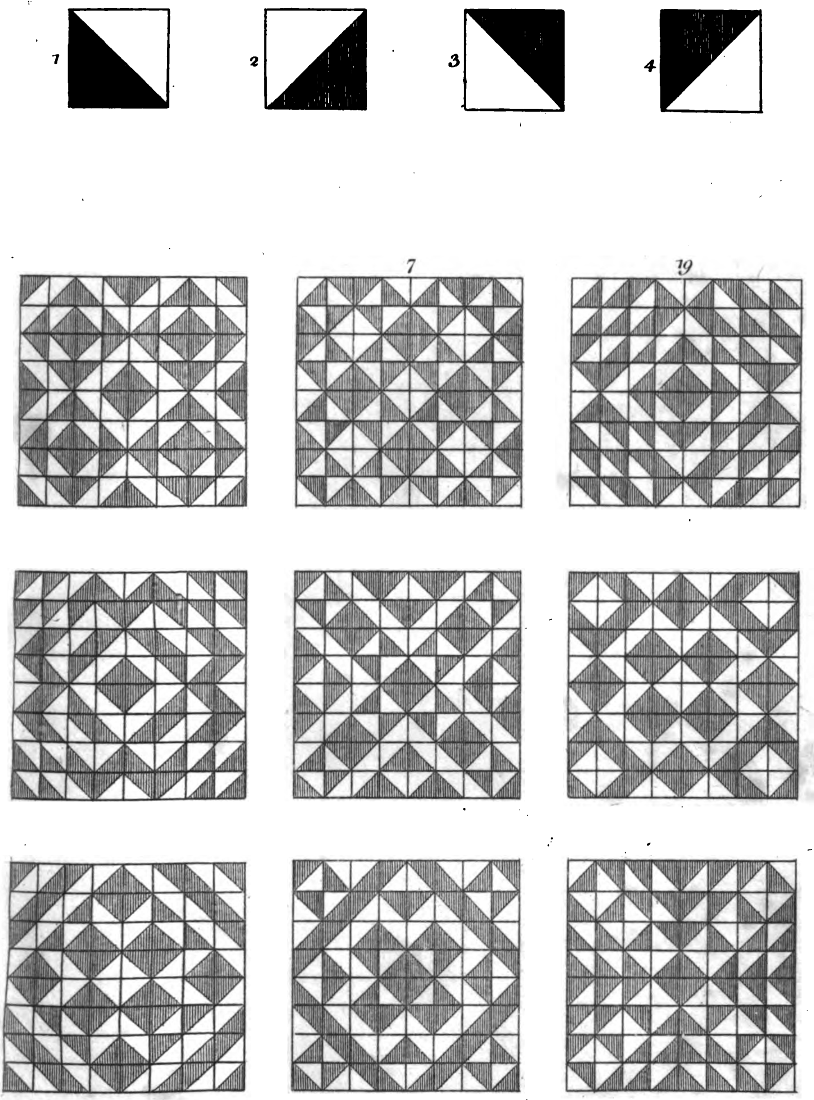


Fig. 6.

Mellor Sc. Bagnell Co.



Mutlow Sc. Russell Co.

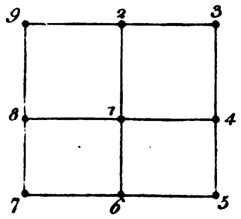


Fig. 1.

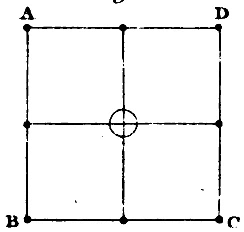
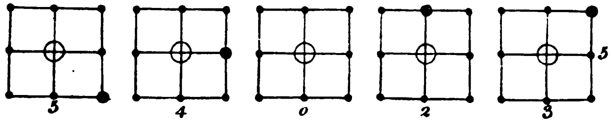
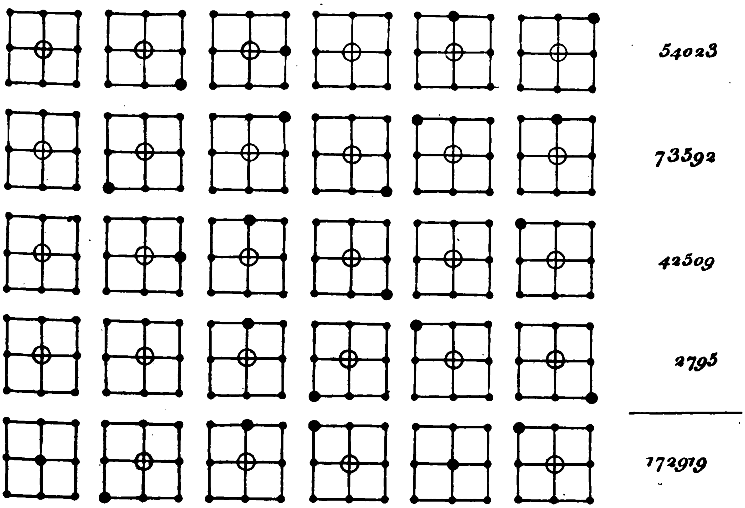


Fig. 2.





Mutlow Sc. Russell Ct.

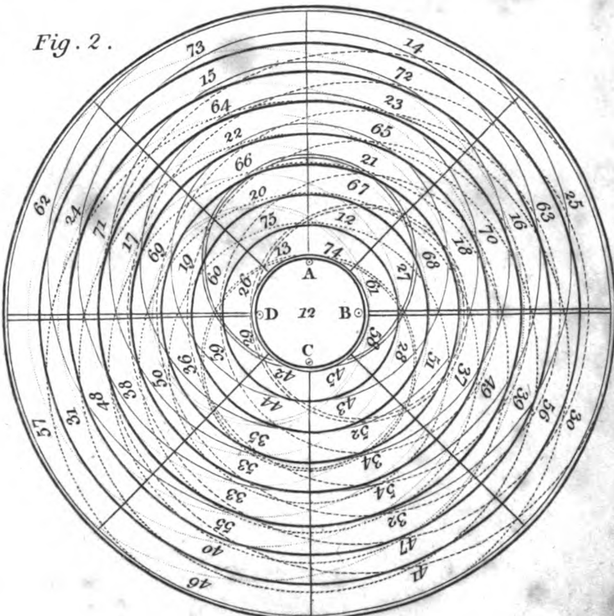
Fig. 1.

MAGIC Square of Squares.

200	217	232	249	8	25	40	57	72	89	104	121	136	153	168	185
58	39	26	7	250	231	218	199	186	167	154	135	122	103	90	71
198	219	230	251	6	27	38	59	70	91	102	123	134	155	166	187
60	37	28	5	252	229	220	197	188	165	156	133	124	101	92	69
201	216	233	248	9	24	41	56	73	88	105	120	137	152	169	184
55	42	23	10	247	234	215	202	183	170	151	138	119	106	87	74
203	214	235	246	11	22	43	54	75	86	107	118	139	150	171	182
53	44	21	12	245	236	213	204	181	172	149	140	117	108	85	76
205	212	237	244	13	20	45	52	77	84	109	116	141	148	173	180
51	46	19	14	243	238	211	206	179	174	147	142	115	110	83	78
207	210	239	242	15	18	47	50	79	82	111	114	143	146	175	178
49	48	17	16	241	240	209	208	177	176	145	144	113	112	81	80
196	221	228	253	4	29	36	61	68	93	100	125	132	157	164	189
62	35	30	3	254	227	222	195	190	163	158	131	126	99	94	67
194	223	226	255	2	31	34	63	66	95	98	127	130	159	162	191
64	33	32	1	256	225	224	193	192	161	160	129	128	97	96	65

Fig. 2.

MAGIC Circle of Circles.



Madlow Sc. Russell Court.

221	212	45	36	205	196	61	52	157	148	109	100	141	132	125	116
35	46	211	222	51	62	195	206	99	110	147	158	115	126	131	142
220	213	44	37	204	197	60	53	156	149	108	101	140	133	124	117
38	43	214	219	54	59	198	203	102	107	150	155	118	123	134	139
215	218	39	42	199	202	55	58	151	154	103	106	135	138	119	122
41	40	217	216	57	56	201	200	105	104	153	152	121	120	137	136
210	223	34	47	194	207	50	63	146	159	98	111	130	143	114	127
48	33	224	209	64	49	208	193	112	97	160	145	128	113	144	129
253	244	13	4	237	228	29	20	189	180	77	68	173	164	93	84
3	14	243	254	19	30	227	238	67	78	179	190	83	94	163	174
252	245	12	5	236	229	28	21	188	181	76	69	172	165	92	85
6	11	246	251	22	27	230	235	70	75	182	187	86	91	166	171
247	250	7	10	231	234	23	26	183	186	71	74	167	170	87	90
9	8	249	248	25	24	233	232	73	72	185	184	89	88	169	168
242	255	2	15	226	239	18	31	178	191	66	79	162	175	82	95
16	1	256	241	32	17	240	225	80	65	192	177	96	81	176	161

Mutlow Sc. Russell Ct.

MATHEMATICAL

AND

PHILOSOPHICAL

RECREATIONS.

PART SECOND.

Containing a Series of Geometrical Problems and Questions, calculated for Exercise and Amusement.

PROBLEM I.

From the extremity of a given right line to raise a perpendicular, without continuing the line, and even without changing the opening of the compass, if necessary.

I. LET AB (fig. 1 plate 1), be the given straight line, and A the extremity, from which it is required to raise a perpendicular, without prolonging it.

From A towards B assume 5 equal parts at pleasure; and extending the compasses from A, so as to include 3 of these parts, describe the arc of a circle; then from *b*, the extremity of the fourth part, with an opening equal to the 5 parts, describe another; these two arcs will necessarily cut each other in a certain point C, from which if a straight line, as CA, be drawn, it will be perpendicular to AB.

For the square of CA, which is 9, added to the square of

A*b*, which is 16, are together equal to 25 the square c*b*: the triangle CA*b* is therefore right angled at A.

We might assume also, for the radius of the ark to be described from the point A, a line equal to 5 parts; for the base A*b* 12, and for the other radius *b*c 13; because 5, 12, and 13, form a right-angled triangle. Indeed, all the right-angled triangles in numbers, of which there are a great variety, may be employed in the solution of this problem.

II. On any part whatever of the given line AB (fig. 2 pl. 1), describe an isosceles triangle ACB, that is, so that the sides AC, CB, shall be equal; and continue AC to D, so that CD shall be equal to CB; if a line be then drawn from D to B, it will be perpendicular to AB. The demonstration of this is so easy, that it requires no illustration.

PROBLEM II.

To divide a given straight line into any number of equal parts, at pleasure, without repeated trials.

Let it be proposed, for example, to divide the line AB (fig. 3 pl. 1), into 5 equal parts. Make this given line the base of an equilateral triangle ABC; and from the point C, in the side CB, continued if necessary, set off 5 equal parts, which we shall suppose to terminate at D, and make CE equal to CD; then make DF, for example, equal to one of the five parts of CD; and draw CF, which will intersect AB in G: it is evident that BG will be the fifth part of AB.

If D*f* were equal to $\frac{2}{5}$ of CD, by drawing C*f* we should have *g*, as the point of intersection of C*f* and AB, which would give B*g* equal to $\frac{2}{5}$ of AB. And so on.

PROBLEM III.

Without any other instrument than a few pegs and a rod, to perform on the ground the greater part of the operations of geometry.

It is well known that most geometrical operations may be performed in the field, by means of the graphometer;

and it would even seem that this instrument is absolutely necessary in practical geometry.

A geometrician however may happen to be unprovided with such an instrument, and even destitute of the means of procuring one. We shall suppose him in the woods of America, with nothing but a knife to cut a few pegs, and a long stick to serve him as a measure: he has several geometrical operations to perform, and even inaccessible heights to measure; how must he proceed to accomplish what is here proposed?

We shall suppose also, that the reader is acquainted with the method of tracing out a straight line on the ground, between two given points; and in what manner it may be indefinitely continued on either side, &c. This being premised, we shall now proceed to give a few of those elementary problems of geometry, required to be performed, without employing any other line than a straight one, and even excluding the use of a cord, with which the arc of a circle might be described.

1st. *Through a given point to draw a straight line, parallel to a given straight line.*

Let AB (fig. 4 pl. 1) be the given straight line, and C the point through which it is required to draw a straight line parallel to AB. From the point C draw the line CB, to any point in AB, and divide CB into two equal parts in D; in this point D fix a peg, and from any point A in the given straight line draw, through D, an indefinite line ADE, and make DE equal to AD: if a straight line be then drawn through the points C and E, it will be parallel to AB.

2d. *From a given point in a given straight line, to raise a perpendicular.*

Divide the given line AB (fig. 5 pl. 1) into two equal parts, AC and CB; and from the point C draw, any how at pleasure, the line *cd;* make CD equal to CA; draw DA*h*, and make AE equal to AC, and AF equal to AD; through the points E and F draw the line FEG; and if EG be made equal to EF, we shall have the point G, which with the point A will determine the position of the perpendicular AG.

For the sides AD and AC of the triangle CAD, being respectively equal to the sides AF and AE of the triangle EAF, these two triangles are equal; and, in the triangle DCA, the sides CD and CA being equal, the sides EA and EF of the other will be equal also: the angle EFA therefore will be equal to EAF, and consequently to CAD. But in the triangle FGA, the side FG is equal to AB, for FG by construction is the double of FE, and FE or AE is equal to AC, which is the half of AB: the triangles FAG and ADB then are equal; since the sides FG, FA are equal to the sides AB, AD, and the included angles equal: the angle FAG will therefore be equal to ADB; but the latter is a right angle, because the lines CB, CD, CA being equal, the point D is in the circumference of a semicircle, described on the diameter AB. The angle FAG then is a right angle, and GA is perpendicular to AB.

3d. *From a given point* A, *to draw a straight line perpendicular to a given straight line.*

Assume any point B (fig. 6 pl. 1) in the indefinite line BC; and having measured the distance BA, make BC equal to BA; draw CA, which must be measured also, and then form this proportion: as BC is to the half of AC, so is AC to a fourth proportional, which will be CE; if CE be then made equal to this fourth proportional, we shall have the point E, from which if the line AE be drawn through A, it will be the perpendicular required.

4th. *To measure a distance* AB, *accessible only at one of its extremities, as the breadth of a river or ditch, &c.*

First fix a peg at A (fig. 7 pl. 1); then another in any point C, assumed at pleasure, and a third at D, in the straight line between the points B and C; continue the lines CA and DA indefinitely beyond A, and make the lines AE and AF respectively equal to AC and AD; in the last place, fix a peg at G, in such a manner as to be in a straight line with A and B, and also with F and E; the distance AG will then be equal to AB.

If it be found impossible to proceed far enough from the line AB towards E or F, we may take in AE or AF only the half or the third of AC and AD, for example A*e*, A*f*; if a peg be then fixed in *g*, so as to fall in the continuation of both the lines BA and *ef*, we shall have A*g* equal to the half or the third of AB respectively.

Now let the distance AB (fig. 8 pl. 1) be inaccessible throughout. The solution of this case may be easily deduced from that of the former: for having fixed a peg in C, and having continued by a series of pegs the lines BC and AC, if the parts CE and CF be, by the above means, made respectively equal to BC and CA, or the half or the third of these lines, it may be readily seen that the line which joins the points E and F, will be equal to the line required, or to the half or third of it; and that in either case it will be parallel to it, which resolves the problem, to draw a line parallel to an inaccessible line.

These examples are sufficient to show in what manner a person, who has only a slight knowledge of geometry, may execute the greater part of geometrical operations, without any other instruments than those which might be procured in a wood by means of a knife. It must indeed be allowed that one can never be in such circumstances, unless on some very extraordinary occasion; but, however, it may afford satisfaction to those who have a turn for geometry, to know in what manner they might proceed, if ever such a case should happen.

It is remarkable, that it is not perhaps possible to resolve in this manner, that is to say without employing the arc of a circle, the very simple problem, and one of the first in the elements of geometry, viz. to describe an equilateral triangle. We have often attempted it, but without success, while trying how far we could proceed in geometry by the means of straight lines only.

PROBLEM IV.

To describe a circle, or any determinate arc of a circle, without knowing the centre, and without compasses.

To those who are little acquainted with geometry, this will appear to be a sort of paradox; but it may be easily explained by that proposition, in which it is demonstrated, that the angles whose summits touch the circumference, and whose sides pass through the extremity of the chord, are equal.

Let A, C, B (fig. 9 pl. 1) be three points in the required circle or arc: having drawn the lines AC and CB, make an angle equal to ACB of any solid substance, and fix two pegs in A and B; if the sides of the determinate angle be then made to slide between these pegs, the vertex or summit will describe the circumference of the circle. So that if the summit or vertex be furnished with a spike or pencil, it will trace out, as it revolves between A and B, the required arc.

If another angle of the like kind were constructed, forming the supplement of ACB to two right angles, and if it were made to revolve with its sides always touching the points A and B, but with its summit in a direction opposite to C, it would describe the other segment of the circle, which with the arc ACB would make up the whole circle.

It may sometimes happen that it is necessary to describe, through two given points, the arc of a determinate circle, the centre of which is at a great distance, or inaccessible on account of some particular causes. Should it be required, for example, to describe on the ground a circle, or the arc of a circle, with a radius equal to 2 or 3 or 4 hundred yards; it may be readily seen that it would be impracticable to do it by means of a cord: the mode of operation therefore must be as follows. In A and B, (fig.

10 pl. 1) the extremities of that line which we here suppose to be the chord of the required arc, the amplitude or subtending angle of which is known, fix two pegs, and then find out, by means of a graphometer or plane table, any point *c*, in such a position, that A*c* and B*c* shall form an angle, A*c*B, equal to the given angle, and in that point fix a peg; then find out another point *d*, so situated that A*d* and B*d* shall form an angle, A*d*B, equal to the former; if the points *f* and *e* be found in like manner, it is evident that the points *c*, *d*, *e* and *f* will be in the arc of a circle capable of containing the given angle. If the points *g*, *h*, *i*, *k*, be then found, on the other side of AB, so situated, that the angle A*g*B or A*h*B, &c, shall be the supplement of the former, the points *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, will evidently be all in a circle.

PROBLEM V.

Three points, not in the same straight line, being given, to describe a circle which shall pass through them.

Let the three points be those marked 1, 2, 3, (fig. 12 pl. 2): from one of them as a centre, that for example marked 2, and with any radius at pleasure, describe a circle; and from one of the other two points, 1 for example, assumed as a centre, make with the same radius two intersections in the circumference of the first circle, as at A and B; draw the line AB, and assuming the third point 3 as a centre, make with the same radius two more intersections in the circumference of the first circle, as D and E: if DE be then drawn, it will cut the former line AB in the point C, which will be the centre of the circle required. If a circle therefore be described from this point as a centre, through one of the given points, its circumference will pass through the other two.

It may be readily seen that this construction is the same, in principle, as the common one, taught by Euclid and all other elementary writers; for it is evident that the lines

1A, 2A, 1B, 2B are equal to each other; consequently the line AB is perpendicular to that which would join the points 1 and 2, or to the chord 1 2 of the required circle; hence it follows, that the centre of the circle is in the line AB: for the same reason this centre is in the line DE, and therefore it is in the point where they intersect each other.

If the three given points were in a straight line, the lines AB and DE would become parallel, and consequently there would be no intersection.

PROBLEM VI.

An engineer, employed in a survey, observed from a certain point the three angles formed by three objects, the positions of which he had before determined: it is required to determine the position of that point, without any farther operation.

This problem, reduced to an enunciation purely geometrical, might be proposed in the following manner: a triangle, the sides and angles of which are known, being given, to determine a point from which, if three lines be drawn to the three angles, they shall form with each other given angles.

In this problem there are a great number of cases; for either the three angles, under which the distances of the three given points are perceived, occupy the whole extent of the horizon, that is to say are equal to four right angles, or occupy only the half, or less than the half. In the first case, it is evident that the required point is situated within the given triangle; in the second it is situated in one of the sides, and in the third it is without. But for the sake of brevity we shall here confine ourselves to the first case.

Let it be required then to determine, between the points A, B, C (fig. 11 pl. 2), the distances of which are given, a point D so situated, that the angle ADB shall be equal to 160 degrees, CDB to 130°, and CDA to 70°. On the side AB describe an arc of a circle capable of containing an angle of 160°; and on the side BC another capable of containing an angle of 130°; the point where they intersect each other will be the point required.

For it is evident that this point is in the circumference of the arc described on the side AB, and capable of containing an angle of 160°; because from all the points of that arc, and of no other, the distance AB is seen under an angle of 160°. In like manner the point D must be found in the arc described on the side BC, and capable of containing an angle of 130°; consequently it must be in the place where they intersect each other, and no where else.

REMARK.

On this construction, a trigonometrical solution may be founded, to determine in numbers the distance between D and the points A, B, and C; but we shall leave this to the ingenuity of the reader.

PROBLEM VII.

If two lines meet in an inaccessible point, or a point which cannot be observed, it is proposed to draw, from a given point, a line tending to the inaccessible point.

Let the unknown and inaccessible point be O (fig. 13 pl. 2), the lines tending to it AO and BO; and let E be the point from which it is required to draw a straight line tending towards O.

Through the point E draw any straight line EC, intersecting AO and BO in the points D and C; and through any point F, assumed at pleasure, draw FG parallel to it; then make this proportion, as CD is to DE, so is FG to GH; if the indefinite line HE be then drawn, through the points E and H, it will be the line required.

Or if the given point be *e*, make this proportion, as CD is to C*e*, so is FG to F*h*; the line *eh* will be that required.

The demonstration of this problem will be easy to those who know, that, in any triangle, if lines be drawn parallel to the base, all those drawn from the vertex of the triangle will divide them proportionally.

PROBLEM VIII.

The same supposition being made; to cut off two equal portions from the lines BO *and* AO (fig. 14 pl. 2).

From the point A, draw AC perpendicular to BO, and AD perpendicular to AO; if the angle CAD be then divided into two equal parts by the line AE, meeting BO in E, this line will cut off from BO and AO the two equal parts, AO and EO.

This may be easily demonstrated, by showing that, in consequence of this construction, the angle OAE becomes equal to OEA. But the angle OAE is equal to the angle OAC plus CAE; and the angle OEA is equal to ODA or OAC plus EAD, or EAC, which is equal to it; the angle OAE then is equal to OEA, and the triangle OAE is isosceles, therefore, &c.

PROBLEM IX.

The same supposition still made; to divide the angle AOE *into two equal parts,* (fig. 14 pl. 2).

Construct the same figure as in the preceding problem; then between the two given lines draw any line FG, parallel to the line AE; and divide the lines AE and FG into two equal parts in H and I: the line HI will divide the angle AOE into two equal parts. The demonstration of this is so easy, that it requires no illustration.

These problems, as may be readily seen, contain operations of practical geometry of great utility in certain cases; such, for example, as when it is necessary to cut roads through a forest, or when it is required to make them tend to, or end at, a common centre.

PROBLEM X.

Two sides of a triangle, and the included angle, being given; to find its area.

Multiply one of the sides by half the other, and the product by the sine of the included angle: this new product will be the area.

It may be easily demonstrated, that the area of every triangle is equal to half the rectangle of any two of its sides, multiplied by the sine of the included angle.

Let ABC (fig. 15 pl. 2), be a triangle, having an acute angle at A; produce AC towards *d*, and from A as a centre, with the distance AB, describe the semicircle BF*b*; then from the point A draw FA perpendicular to AC; and from the point B draw BD also perpendicular to AC.

It is here evident that the two triangles FAC and BAC are respectively to each other as AF is to BD; that is to say, as radius is to the sine of the angle BAC, or as unity is to the number which expresses that sine; the triangle FAC then being equal to half the rectangle of FA by AC, the other will be equal to that half rectangle multiplied by the sine of the angle BAC.

This property enables us to avoid that tedious process, necessary to be employed in order to find the measure of the perpendicular let fall from the extremity of one of the known sides on the other, that the latter side may be then multiplied by the half of this perpendicular.

Thus, for example, let the two sides AB and AC be respectively equal to 24 and 63 yards; and let the included angle be 45°. The product of 63 by 12 is 756, and the sine of 45° is 0·70710; if 756 therefore be multiplied by 0·70710, according to the method of decimal fractions, the product will be $534\frac{56}{100}$.

PROBLEM XI.

To find the superficial content of any trapezium or quadrilateral figure, without knowing its sides.

The solution of this problem is a consequence of the preceding. Let the given trapezium be ABCD (fig. 16 pl. 2); measure the diagonals AC and BD, as well as the angle which they make at the point where they intersect each other in E; if these diagonals be then multiplied together, and half their product by the sine of the above angle, the last product will be the area. This method is far shorter than if we should reduce the trapezium to triangles, in order to find the area of each of them.

COROLLARIES.

A very curious theorem, which no author has before remarked, may be deduced from this problem. It is as follows: If two quadrilateral figures have their diagonals equal, and intersecting each other at the same angle, whatever may be their difference in other respects, these quadrilateral figures will be equal as to their area.

1st. Thus, the quadrilateral ABCD (fig. 16), is equal to the parallelogram *abcd* (fig. 17 n°. 1), which has its diagonals equal to those of ABCD, and inclined towards each other at the same angle.

2d. The same quadrilateral ABCD, is equal to the triangle BAC (fig. 17 n°. 2), formed by the two lines AC and AB, equal to the diagonals AC, BD, and inclined at the same angle.

3d. The same quadrilateral will be equal also to the triangle ABC (fig. 17 n°. 3), if the lines AC and DB of that triangle are equal to the diagonals of the quadrilateral, and equally inclined.

4th. In the last place, this same quadrilateral ABCD (fig. 16), will be equal to the quadrilateral *abcd* (fig. 17 n°. 4), the diagonals of which do not intersect each other, if *ac* and *db* are equal to AC and DB, and if the angle *bec* is equal to the angle BEC.

PROBLEM XII.

Two circles, not entirely comprehended one within the other, being given; to find a point from which, if a tangent be drawn to the one, it shall be a tangent also to the other.

Through the centres A and B (fig. 18, n°. 1, pl. 3), of the two circles, draw the indefinite straight line ABI: then from the centre A draw any radius AC, and through the centre B draw the radius BD parallel to it. If the points C and D be joined by the line CD, it will meet AB in I, which will be the point required; that is to say, if IE be drawn from the point I a tangent to one of the circles, it will be a tangent also to the other.

When the circles do not cut each other, the point I (fig. 18, n°. 2) may happen to fall between them. To find it, in that case, nothing is necessary but to draw the radius BD parallel to AC, and in a direction opposite to that of fig. 18 n°. 1. AB and CD will then intersect each other in the point I, which will have the same property as the former.

REMARK.

We cannot here help observing, that if any secant whatever, as IDH or I*dh* (fig. 18 n°. 1), be drawn from the point I, through the two circles, the rectangle of ID and IH, or of I*d* and I*h*, will be always the same, that is, equal to the rectangle of the two tangents IE and IF. In like manner, the rectangle of IC and IG, or of I*c* and I*g*, will be equal to the rectangle of the same tangents. This is a very remarkable extension of the well known property of the circle, by which the rectangle of the two segments ID and IG is equal to the square of the tangent IE.

PROBLEM XIII.

A gentleman, at his death, left two children, to whom he bequeathed a triangular field, to be divided equally between them; in the field is a well, which serves for watering it; and as it is necessary that the line of division should pass through this well, in what manner must it be drawn, so as to intersect the well, and divide the field, at the same time, into two equal parts?

Let the given triangle be ABC (fig. 19 pl. 3), and the given point be E. From the point E draw the lines ED and ER, parallel to the base and the side BC respectively, and meeting them in D and R; let the base AC be divided into two equal parts in M; and having drawn the line DM from the point D, draw BN parallel to it, and divide CN into two equal parts, in I; on IR describe the semicircle IKR, in which apply RK = RC; and, having drawn IK, if IF be made equal to it, the points F and E will determine the line FEG.

REMARK.

It is evident that CI must be at least double of CR; otherwise CR could not be applied in the semicircle described on IR, which would render the problem impossible.

In numbers. Let AB = 48 fathoms, BC = 42, AC = 30, CD = 18, and DE or CR = 6; consequently CM will be = 15. But CD : CM :: CB : CN, that is to say 18 : 15 :: 42 : 35; hence it follows that CN = 35, and CI = $17\frac{1}{2}$; and as CR is equal to 6, we shall have IR = $11\frac{1}{2}$. But the triangle IKR being right-angled, IK = $\sqrt{IR^2 - RK^2}$ = $\sqrt{132\frac{1}{4} - 36}$ = $\sqrt{96\frac{1}{4}}$, or $9\frac{81}{100}$ fathoms, which gives CF = $27\frac{31}{100}$ fathoms.

The demonstration of this construction is too prolix to be given in this work; and there are even a variety of cases, which it would be tedious to explain. We shall therefore confine ourselves to one of the simplest; that is, where the point E is in one of the sides (fig. 20 pl. 3).

The construction in this case is exceedingly easy; for having divided AC into two equal parts in M, and drawn EM, and BN parallel to it: if the point N falls within the triangle, by drawing the line EN, the problem will be solved; but if the point N falls without the triangle, it will be necessary to draw the line AE; then NO parallel to it, through the point N, and OE through the point O: the last line, OE, will solve the problem.

For because EM is parallel to BN, the triangle MBE = MNE; and if the triangle CME be added to each, we shall have the triangles CBM and CEN equal to each other. But the triangle CBM is the half of the triangle ABC, because AM = MC; consequently CEN is the half of ABC also. In like manner, because EA is parallel to NO, the triangles ANE and AOE are equal; and therefore if the triangle AGE, which is common to both, be taken away, the triangle ANG will be equal to GOE; hence it follows, that if we add to the space CAGE the triangle GOE, we shall have the space CAOE = the triangle CEN, which we have already shown to be equal to the half of ABC.

But if the gentleman had left the field to be divided equally among three children, by lines proceeding from the given point E (fig. 21 pl. 3); if we suppose one line of division EB already drawn, it would be necessary to proceed as follows:

Divide the base AC into three equal parts; and let the points of division be D and G; draw the line ED, and BF parallel to it; then draw the line EF from the point E, and if the point F does not fall without the triangle, the trapezium BEFAB will be one of the thirds required.

But if the point F falls without the triangle, we must proceed as above directed; that is to say, the line EA must be drawn to the angle A, and FO parallel to it from the point F, as far as the side AB, which it meets, we shall suppose, in O; the line EO will give the triangle BOE, equal to the third of the triangle proposed. BEICB, the other third, may be found in like manner; consequently the remainder of the figure will be a third also. The three lines therefore, EO, EI, and EB, which proceed from the point E, will divide the proposed triangle into three equal parts.

By the same method a triangle might be divided into 4, or 5, or 6, &c, equal parts, by lines all proceeding from a given point; and this point may be assumed even without the triangle.

PROBLEM XIV.

Two points and a straight line, not passing through them, being given: to describe a circle which shall touch the straight line, and pass through the two given points.

Let the given line be AB (fig. 22 pl. 3), and the given points C and D. Join these two points, and on the middle of the line CD raise the perpendicular EF, meeting the given straight line in F; and on the same line let fall the perpendicular EH; draw FC, and from the point E, with the radius EH, describe a circle intersecting FC, continued, in I; draw IE, and through the point C draw CK parallel to it: the point K will be the centre, and KC the radius of the circle required.

For if the perpendicular KL be let fall from the point K, on the line AB, it will be equal to KC, which is equal to KD. But FE is to FK, as EH is to KL, and as EI to KC; therefore EH is to KL as EI to KC; and consequently, as EI is equal to EH, KL will be equal to KC; therefore, &c.

It may be readily seen, that if the given line passed through one of the given points, the centre of the required circle would be in the point K (fig. 23 pl. 3), where CK, drawn perpendicular to AB, intersects EK, which is perpendicular to CD, and divides it into two equal parts in E.

In the first case, the problem might be resolved in a different manner, viz, by continuing the line CD (fig. 22), till it meets AB in M; then taking a mean proportional between MC and MD, and making ML equal to it; if a circle were then described through the points C, D, L, it would be the one required. But this solution would be attended with difficulty, if the point M were at a great distance, whereas in the former case this is a matter of indifference.

PROBLEM XV.

Two lines AB *and* CD (fig. 24 pl. 3), *with a point* E *between them, being given; to describe a circle which shall pass through this point, and touch the two lines.*

If the two lines meet, as at F, draw the line FH dividing the angle BFD into two equal parts; or, if they are parallel, draw one, such as FH (fig. 25), equally distant from both; then from the point E draw EGI, perpendicular to FH, and make GI equal to GE; the points I and E will be so situated, that if a circle, touching one of the given lines, be described through them, it will touch the other given line also; which reduces this problem to the preceding one.

THEOREM I.

Various demonstrations of the forty-seventh proposition of the first book of Euclid, by the mere transposition of parts.

The beauty of this elementary proposition, and the difficulty beginners often find to comprehend the demonstration, have induced some geometricians to invent others of a simpler nature. Some of these are very ingenious, and worthy of notice, because it can be seen on the first view, that the square of the hypothenuse is composed of the same parts as the squares of the two sides: with this difference only, that they are differently arranged. Some of these demonstrations are as follow:

1st. Describe the right-angled triangle ABC (fig. 26 pl. 4), and on the two sides of it, AC and BC, construct the two squares CG and CD. On the base AB raise the two perpendiculars AI and BH, the former meeting GF continued, in I, and the latter meeting ED in H; and then draw IH. It may, in the first place, be easily demonstrated that AI and BH are equal to AB; so that AIHB is the square of the base AB; for it may be readily seen that the triangle

BHD is equal and similar to the triangle BAC, as well as the triangle IGA; so that BH and AI are each equal to AB.

It may be shown, with equal ease, that the small triangle KEH is equal to IFO; and lastly that the triangle IKL is equal to AOC.

But the constituent parts of the two squares are, the quadrilateral CBHK, the triangle BDH, the triangle KHE, the quadrilateral GAOF, and the triangle ACO, which we shall show to be the same that compose the square ABHI; for the quadrilateral CBHK is common, and the triangle BHD is equal to BCA, and may be substituted for it, and transposed into its place. In like manner, we may conceive the triangle ACO transposed into IKL; there will then remain, in the square of the hypothenuse, the vacuity ILA, and we shall have, to fill it up, the quadrilateral FOAG, with the triangle KEH: let the triangle KEH be transposed into OFI, which is equal to it, and it will complete the triangle IAG, which is equal and similar to IAL; hence it follows that the square of the hypothenuse is composed of the same parts as the squares of the other two sides.

We may therefore cut these parts from a piece of card, and first compose the two squares of the two sides, and then that of the hypothenuse, which will form a sort of amusement in combination.

2d. The second method, which is nearly the same as the preceding, will appear perhaps a little more evident. Let CD and CF (fig. 27) be the squares of the two sides, which contain the right angle of the triangle ACB: having continued FA until AH is equal CB, on the side FH construct the square FHDG: and on AB, the hypothenuse, the square AE. It may be easily proved that the angles E and N will be in the sides of the former, and that AH, BD, EG, NF will be all equal, as well as FA, BH, DE, GN.

But it may be readily seen that, by drawing the line NI parallel to FH, the two squares CD and CF will be composed of the parts 1, 2, 3, 4, 5; and the square AE is composed of the parts 1, 5, 6, 7, 8. But the parts 1 and 5 are common, and the parts 6 and 2 are evidently equal: it remains then that the parts 4 and 3 should be equal to the parts 7 and 8. But this is also evident; for the part 3 is equal to 9, and the part 8 to 5, consequently the parts 4 and 3, or 4 and 9, are equal to the parts 7 and 8, or 7 and 5, since the rectangle FI is divided into two equal parts by the diagonal. The squares of the sides then are composed of the same parts as the square of the hypothenuse, and consequently they are equal.

3d. Retaining the same construction, it is evident that the square FD is equal to the squares of the two sides AC and CB of the right-angled triangle ACB, plus the two equal rectangles CG and CH. But the square AE, of the hypothenuse, is equal to the same square less the four equal triangles ABH, BED, EGN, NFA, which taken together are equal to the two rectangles above mentioned, since each of the triangles is the half of one of the rectangles. The quantity by which the square FD exceeds the squares of the sides of the right-angled triangle ACB, is the same as that by which it exceeds the square of the hypothenuse: these squares and that of the hypothenuse are therefore equal; for quantities which are less than a third by an equal quantity, are themselves equal.

We shall now give a few propositions which are only generalizations of the forty-seventh of the first book of Euclid, and from which that celebrated proposition is deduced as a simple corollary.

THEOREM II.

If a square be described on each of the sides of any triangle ABC (fig. 28 and 29, pl. 4), *and if a perpendicular* BD *be let fall from one of the angles, as* B, *on the opposite side* AC; *if the lines* BE *and* BF *be drawn in such a manner that the angles* AEB *and* CFB *shall be equal to the angle* B;

and lastly if EI *and* FL *be drawn parallel to* CG, *the side of the square, the square of* AB, *will be equal to the rectangle* AI, *and the square of* BC *to the rectangle* CL; *consequently the sum of the squares on* AB *and* BC *will be equal to the square of the base less the rectangle* EL, *if the angle* B *be obtuse, and plus the same rectangle if the angle* B *be acute* *.

The demonstration of this theorem is as follows: the triangle AEB is similar to the triangle ABC, because the angle A is common, and the angle AEB equal to the angle ABC; consequently AC : AB :: AB : AE, whence it follows that the rectangle of AC × AE, or of AE × AH, which is the same since AH = AC, is equal to the square of AB.

In like manner it may be proved that the square of BC is equal to the rectangle CL.

But it may be readily seen, that if the angle B be obtuse, the line BE will fall between the points A and D, and the line BF between C and D; the contrary of which is the case if the angle B be acute; and that these two lines are confounded with, or coincide with, the perpendicular BD, when the angle B is a right one.

In the first case then it is evident, that the sum of the squares of the sides, is less than the square of the base by the rectangle EL.

And in the second case, that they exceed it by the rectangle EL.

Lastly, that if the triangle be right-angled at B; as the rectangle EL vanishes, the sum of the squares of the sides is equal to the square of the base; which is a very ingenious generalization of the celebrated theorem of Pythagoras.

* For this ingenious theorem, from which is deduced the famous problem of the right-angled triangle, we are indebted to Clairault, junior, who published it at the age of sixteen, in a small work printed in 1731. This young man would certainly have trodden in the steps of his brother, had he not been cut off by a premature death.

THEOREM III.

Let ABC (fig. 30 pl. 4), *be a triangle, and let any parallelogram* CE *be described on the side* AC, *and any parallelogram* BF *on the side* AD; *continue the sides* DE *and* KF *till they meet in the point* H, *from which draw the straight line* HAL, *and make* LM *equal to* HA; *if the parallelogram* CO *be then completed on the base* BC, *by drawing* BO *or* CN *parallel to* LM, *this parallelogram will be equal to the two* CE *and* BF.

Continue OB and NC till they meet the sides of the parallelograms BF and CE, in P and R, and draw PR.

Then since CR and HA are parallel, and comprehended between the same parallels, viz. CA and DH, they are equal; consequently CR is equal to LM. In like manner it may be demonstrated that BP is equal to LM. CR and BP therefore are equal, and the figure BPRC is a parallelogram equal to BN.

Now it is evident that the parallelogram RL, on the base RC, is equal to the parallelogram RCAH, because it is on the same base and between the same parallels; and for the same reason the parallelogram ACDE = ACRH; consequently the parallelogram ACDE = RCLG.

It may be demonstrated in like manner that the parallelogram BKFA = BPGL; consequently the two parallelograms CE, BF are together equal to BPRC or to BCNO, which is equal to it.

COROLLARY.

The reader, if in the least acquainted with geometry, may readily see, that this very ingenious proposition is only a generalization of the celebrated proposition, by which it is proved, that in every right-angled triangle, the squares of the two sides, containing the right angle, are equal to that of the hypothenuse. For if we suppose that the triangle BAC is right-angled at A, and that the two parallelograms CE and BF are the two squares, it may be easily conceived that the third parallelogram BN will be also a square, viz. that of the hypothenuse; in consequence of the preceding demonstration then, these two first squares will be equal to the third. This theorem is extracted from Pappus Alexandrinus.

THEOREM IV.

In every parallelogram, the sum of the squares of the four sides, is equal to the sum of the squares of the two diagonals.

Let ABCD (fig. 31 pl. 4), be an oblique parallelogram, the diagonals of which are AD and BC. From one of the angles A let fall, on the diagonal CB, the perpendicular AF; and by Euclid, book II, prop. 12, the square of AB will be equal to the square of AE, plus the square of BE plus twice the rectangle of FE and EB: the square of AC also will be equal to the sum of the squares of AE and EC, minus twice the rectangle of FE and EC, which is equal to that of FE and EB, because EB is equal to EC: the sum of the squares of AB and AC then is equal to twice the square of AE, plus the square of EB, plus the square of EC, or twice the square of AE plus twice that of BE.

But the squares of BD and DC are equal to the squares of AB and AC, because the lines CD and BD are respectively equal to AB and AC; the four squares of the four sides, therefore, are equal to four times the square of BE plus four times that of AE. But four times the square of BE, is equal to the square of BC, and four times the square of AE is equal to the square of AD: therefore, &c.

We shall terminate this series of theorems with the following one, respecting any kind of quadrilateral figures whatever.

THEOREM V.

In every quadrilateral figure whatever, the sum of the squares of the four sides, is equal to the sum of the squares *of the two diagonals, plus four times the square of the line which joins the middle of these diagonals.*

Let ABCD (fig. 32, pl. 4) be a quadrilateral figure, the two diagonals of which are AC and BD; and let us suppose them divided each into two equal parts in E and F, and that the straight line EF has been drawn. It may be demonstrated, that the squares of the four sides, taken together, are equal to the squares of the two diagonals, plus four times the square of EF.

We shall confine ourselves here to the enunciation of this elegant and very curious problem, for which, it is said, we are indebted to the celebrated Euler. The demonstration of it, which is too prolix to be admitted into this work, may be found in the New Memoirs of the Academy of Petersburg, vol. I.

We shall only observe that, when the quadrilateral ABCD becomes a parallelogram, the two diagonals then equally intersect each other, which makes the points E and F to coincide, and by these means the line EF vanishes. The preceding theorem, therefore, is only a particular case of the present one.

PROBLEM XVI.

The three sides of a rectilineal triangle being given; to determine its superficial content, without measuring the perpendicular let fall from one of the angles on the opposite side.

From half the sum of the three sides subtract each of the three sides separately; multiply the three remainders together, and the product by the half sum of the sides; if the square root of the last product be then extracted, it will be the area required.

Let the three sides, for example, be 50, 120, and 150 yards; the half sum of which is 160; the first difference is 110, the second 40, and the third 10: the product of these four numbers is 7040000, the square root of which is 2653 and $\frac{3}{10}$ nearly, which is the area.

It might easily be shown, that the usual method, that is to say, by finding the perpendicular let fall from one of the angles on the opposite side, would require a much more tedious calculation.

REMARK.

By this method we have a very easy rule for finding the radius of the circle inscribed in a triangle, the three sides of which are given: nothing is necessary but to multiply together the difference between each side and the half sum; to divide the product by this half sum, and to extract the square root of the quotient: the result will be the radius required.

Thus, in the above example, the product of the differences is 44000; which divided by 160, gives 275; the square root of this quotient $16\frac{58}{100}$, is the radius of the circle inscribed in the given triangle.

PROBLEM XVII.

In surveying the side of a hill, ought its real surface to be measured, or only the space occupied by its horizontal projection?

It may be easily proved that, in this case, the horizontal projection or base only ought to be measured; for the object of surveying is nothing else than to determine the quantity of any kind of production that land is capable of producing, or the number of the buildings that can be erected on it. But it is evident that as trees and plants always rise in a direction perpendicular to the horizon, an inclined plane can contain no more than the horizontal one which corresponds to it as its base. In like manner, no more buildings can be raised on inclined ground, than on its horizontal projection; because the walls of an edifice must always be vertical: a little more care only is required in building on such ground than on horizontal.

Another reason is, that inclined ground, compared with the horizontal ground in the neighbourhood, contains less vegetable earth or mould, as part of it is always carried away by the rains, and deposited on the lower grounds; consequently it is not capable of supplying nourishment to such a quantity of productions as the other.

It is therefore evident that the horizontal surface only, and not the real or inclined surface, ought to be measured, unless these considerations are thought to be of little value in adjusting the price.

REMARK.

It is in topographical descriptions of mountainous countries chiefly, that care should be taken to reduce the whole to a horizontal plane; for if we suppose that a country has been surveyed, and that, in measuring the sides of pretty steep mountains, the real and not the horizontal distances of places have been taken, it will be impossible, in constructing a map, to make the measures agree. This indeed would be the same thing as if one should attempt to transfer to the plane or base of a pyramid, the triangles which form its inclined sides; for if one of the triangles were laid down on it, all the rest would be falsely represented.

PROBLEM XVIII.

To form one square of five equal squares.

Divide one side of each of four of the squares, as A, B, C, D, (fig. 123 n°. 1 and 2, pl. 15), into two equal parts, and from one of the angles adjacent to the opposite side draw a straight line to the point of division; then cut these four squares in the direction of that line, by which means each of them will be divided into a trapezium and a triangle, as seen fig. 123, n°. 1.

Lastly, arrange these four trapeziums and these four triangles around the whole square E, as seen fig. 123 n°. 2; and you will have a square evidently equal to the five squares given.

REMARK.

By means of the solution to the following problem, one square may be formed of any number of squares at pleasure; for any number of squares may be transformed into an oblong, and we shall show, in the next problem, how an oblong may be resolved into several parts, susceptible of being arranged in such a manner as to form a square.

PROBLEM XIX.

Any rectangle whatever being given; to convert it, by a simple transposition of parts, into a square.

Let the given rectangle be ABCD (fig. 124 pl. 15). To cut it into several parts susceptible of being arranged in a square, first find the geometric mean proportional between the sides BA and AD; make AE equal to that mean proportional, and draw EF perpendicular to AE. EF will cut AD in the point F, which will either fall beyond D, in regard to the point A, or on the point D itself, or between D and A: this forms three cases, the last of which subdivides itself into two, but if one of them be well understood, there will be no difficulty in the rest.

Case 1st. In the first place then, let the point F be beyond D, as seen fig. 124, n°. 1. As the line EF will intersect CD in the point L, make AG equal to DL, and draw GH perpendicular to AE, by which means GH will cut off from the triangle ABE, the small triangle AGH.

Then cut the given rectangle AC into four parts according to the lines AE, EL, and GH, and the result will be the trapezium AELD, the triangle ECL, the trapezium GBEH, and the small triangle AGH, which we shall respectively denote by the letters *a*, *b*, *c*, *d*; lastly, arrange these four parts, as seen fig. 124 n°. 2, and you will have a perfect square.

The demonstration may be easily found, by considering, in fig. 124 n°. 1, the square constructed on AE, viz. AEKI; but it is first necessary to show, that if AI be drawn parallel to EF, and KI, through the point D, parallel to AE, the rectangle AEKI, thence resulting, will be a square. Now this is easy; for if IK be continued till it meet BC produced in P, we shall evidently have the rectangle AEKI equal to the parallelogram AEPD, which is equal to the rectangle ABCD, or that of AB and AD; hence it follows that AE into AI is equal to AB × AD. But the square of AE is equal to AB into AD, consequently AE into AI is the same thing as the square of AE.

This being demonstrated, draw LG parallel to AD, and LM parallel to AE; then, from the points M and G, to AD and AE, draw the perpendiculars MN and GH. It is here evident that the triangle AMN is equal and similar to ELC: in like manner the triangle AGH is equal and similar to DLK; and the trapezium BEHG is equal and similar to NDIM, for BE is equal and parallel to DN, BG to MN, DI to EH, and MI to GH. The four parts AELD, ECL, BEHG, AGH, which compose the rectangle AC, are therefore equal to the four AELD, AMN, NDIM, and DLK, which compose the square AEKI, or, its equal, that of the same figure, n°. 2, &c.

Case 2d. If the point F falls on the point D, the solution of the problem will be exceedingly easy; for in that case the triangle *d* vanishes, since DL vanishes; the square equal to the rectangle, therefore, will be composed of the right-angled isosceles triangle AED (fig. 124 n°. 3), and the other two right-angled and isosceles triangles ABE and CDE, equal to each other, and to the half of the former; consequently these parts may be arranged in a square without any difficulty. This case indeed can never exist but when the side AB is exactly the half of AD: the rectangle AC is then composed of two equal squares. But the manner in which two equal squares may be formed into one is well known.

Case 3d. Let us now suppose that the point F falls between A and D (fig. 125), but in such a manner that FD is less than EB. In this case make EG equal to FD, and draw GH perpendicular to AE; by which means the rectangle AC will be divided into four parts, viz, the triangle AEF, the trapezium CDFE, the trapezium ABGH, and the triangle EGH; which we shall distinguish by the letters, *a*, *b*, *c*, *d*. If these four parts be arranged as seen fig. 125 n°. 2, we shall have a perfect square, as may be easily demonstrated.

If FD be exactly equal to BE, it is evident, that instead of the trapezium ABGH, we should have a triangle AB*h*; so that the square to be formed would consist of three triangles, and a trapezium ECDF, as seen fig. 125 n°. 2.

If FD exceeds EB, and is exactly equal to AF, draw DM parallel to EF, and if the rectangle be cut according to the lines AE, EF, and MD, there will be formed three triangles and a parallelogram ED, which if arranged as seen fig. 125 n°. 3, will compose the square AIKE.

Lastly, we may suppose the height AD (fig. 126 pl. 16), of the given rectangle to be such, that having the general construction described in the first part of this problem, the line FD exceeds the line AF, or is any multiple of it, with or without a remainder. In that case, to resolve the problem, set off the line AF as many times as possible on FD. For the sake of simplification, we shall here suppose that the former is contained in the latter only once with the remainder LD. Draw LM parallel to EF, and by these means we shall have the parallelogram LMEF, which may be placed in FAND; then make EG equal to DL, and draw GH perpendicular to AE; cut the rectangle ABCD according to the lines AE, EF, ML and GH, into five parts, viz, the triangle AEF, the parallelogram FLME, the trapeziums LDCM, AHGB, and the triangle GHE; which we shall distinguish by the letters *a*, *b*, *c*, *d*, *e*; these five parts can be arranged into a perfect square, as AIKE, which is composed of the triangle *a*, the parallelogram *b*, the trapeziums *c* and *d*, and the small triangle *e*.

If AF were contained twice in FD, six parts would be requisite; two of them parallelograms, as *b*.

By a sort of retrograde progress, the following problem may be resolved.

PROBLEM XX.

To cut a given square into 4, or 5, or 6, &c. dissimilar parts, which can be arranged so as to form a rectangle.

Let it be required, for example, to divide the square AEKI (fig. 125, n°. 1, pl. 15), into four parts, susceptible of such an arrangement. On the side EK assume EF greater than the half of it, and draw AF; make AO equal to EF, and draw OM parallel to AF; lastly, from the point M, where OM meets IK, draw MN perpendicular to AF; the four parts required will be the triangles AEF, OMI, and the two trapeziums AOMN, MNFK, which may be arranged in such a manner, as to form the rectangle ABCD. To those who have comprehended the solution of the preceding problem this will appear evident.

If five parts be required, assume EF (fig. 126 pl. 16) of such a length, that it may be contained in EK twice, with a remainder; let these parts of the line EK be EF and FO, and let the remainder be OK; draw AF, and, making AN and NP each equal to EF, draw NO and PQ parallel to AF, the latter of which will meet the side KI in Q; from this point draw QR perpendicular to NO; and we shall have two triangles, a parallelogram, and two trapeziums, which are evidently susceptible of being formed into an oblong such as ABCD; since they are the same parts into which that oblong might be divided, in order to form, by their transposition, the square AEKI: therefore, &c.

PROBLEM XXI.

A transposition, from which it seems to result that a part may be equal to the whole.

Construct a right-angled parallelogram, having its long sides equal to eleven parts, and the short ones to three, and divide it into equal squares, by parallel lines drawn from each of the points of division, as seen fig. 127 n°. 1. By these means we shall have 33 equal and similar squares.

Through the opposite angles draw the diagonal AB, and if the parallelogram be then cut according to the lines EF and GH, and the diagonal BA, we shall have four pieces, which when put together, as seen fig. 127 n°. 1, will contain 33 squares.

But if they be placed together in such a manner, that the line AH (fig. 127, n°. 2 and 3), joins the line BF, and that the two triangles BHG and EFA form a rectangle, we shall have 34 squares instead of 33.

Here then we have 33 equal to 34; but this is only an illusion, which may easily be discovered; for it may be readily seen that all the squares traversed by the oblique lines of union, AH and AB, are each less in height by $\frac{1}{11}$ than the rest. But 11 squares are traversed in this manner, consequently it needs excite no surprise that there should appear to be one more.

This deception, it must be allowed, is very puerile in the eyes of a geometrician; but it is still more ingenious than that of M. G———, for by making the long sides of the rectangle to consist of ten parts, the squares traversed by the line of union want exactly one fifth in height of their breadth, by which means the most inexperienced eye cannot take them for perfect squares, similar to the others; but when they want only an eleventh part of their just dimensions, it is difficult to observe the deficiency.

REMARK.

It was no doubt by a similar deception, that M. Liger pretended to demonstrate, that twice 144 or 288 was equal to 289, the square of 17; from which he concluded that the square of 17 was equal to twice the square of 12, and that 17 was the exact value of the diagonal of a square having 12 for its side. It is hardly possible to believe that any one could be so weak as to maintain such absurdities.

PROBLEM XXII.

To divide a line in extreme and mean ratio.

A line is divided in extreme and mean ratio, when the whole line is to one of the segments, as that segment is to the other. As a great many geometrical problems are reduced to this division, some of the geometricians of the sixteenth century gave it the name of the *divine section.* But without adopting so emphatical a denomination, we shall proceed to the solution of the problem.

Let the line, to be divided in extreme and mean ratio, be AB (fig. 33 pl. 4). From its extremity B raise the perpendicular BC, and make it equal to the half of AB; draw AC, and make CD equal to CB; if AE be then made equal to the remainder AD, the line AB will be divided as required, and we shall have this ratio: AB is to AE as AE is to EB.

REMARKS.

The line *ab* (fig. 34) being divided in extreme and mean ratio, if its greater segment be added to it, we shall have the line *bc*, also divided in extreme and mean ratio, in the point *a*; so that *bc* will be to *ba* as *ba* is to *ac*.

2d. The line *ba* (fig. 34 n°. 2), being divided, in the same manner in *c*, if *cd* be made equal to the small segment *bc*, *ca* will then be divided in the same manner; that is to say, *ca* will be to *cd* as *cd* to *da*.

PROBLEM XXIII.

On a given base to describe a right-angled triangle, the three sides of which shall be in continued proportion.

On the given base AB (fig. 35 pl. 5) describe a semicircle; divide AB in extreme and mean ratio in C, and raise the perpendicular CD till it meet the semicircle in D; then draw the lines AD and DB: the triangle ABD will be the one required, and AB will have the same ratio to AD as AD has to DB, as might be easily demonstrated.

PROBLEM XXIV.

Two men, who run equally well, propose for a bet to start from A, *and to try who shall first reach* B, *after touching the wall* CD, (fig. 36 pl. 5). *What course must be pursued in order to win?*

It may be readily seen that, to determine the course to be pursued in order to win, it will be necessary to determine the position of the lines AE and EB, of such a nature, that their sum shall be less than that of all the others, as A*e*, *e*B, &c. But it may be demonstrated that this sum is the least possible, when the angle AEC is equal to the angle BED. For let us suppose AC drawn perpendicular to CD, and continued till CF be equal to AC, and that EF and EB have been drawn; in this case the angles AEC and CEF will be equal. But AEC is equal to BED by the supposition, consequently the angles CEF and BED will be equal also; and it thence follows, that, as CD is a straight line, FEB will likewise be one. But BEF is equal to BE and EA taken together, as B*e* and *e*F are to B*e* and *e*A; the course BEA therefore will be shorter than any other B*e*A, for the same reason that BF is shorter than the lines B*e* and *e*F.

To find then the point E, we must draw AC and BD perpendicular to the line CD, and then divide CD in E, in such a manner, that CE shall be to ED as CA to DB.

PROBLEM XXV.

A point, a circle, and a straight line, being given in position, to describe a circle which shall pass through the given point, and touch the circle and straight line.

Through the centre of the given circle draw BE (fig. 37 pl. 5) perpendicular to the given straight line, and let it cut the circle in B and F; draw also BA to the given point A, and take BG a fourth proportional to BA, BE, BF; if a circle be then described through the points A and G, touching the line CD, it will touch also the given circle.

If the point A be within the circle (fig. 38), the construction will be the same: in this case it is evident that the line which ought to be touched by the required circle, must enter the given circle also; and there are even two circles which will resolve the problem, as may be seen in fig. 38.

PROBLEM XXVI.

Two circles and a straight line being given; to describe a circle which shall touch them all.

This problem is evidently susceptible of several cases; for the circle which touches the straight line may inclose both the other circles, or only one of them, or may leave them both without it; but, for the sake of brevity, we shall confine ourselves to the last case, and leave the rest to the sagacity of our readers, who, when they comprehend this solution, will find no difficulty to resolve the rest.

Let there be given two circles, whose radii are CA and *ca* (fig. 39 pl. 5), and let the line DE be given in position. In the present case, on the radius CA make AO equal *ca*, and with the radius CO describe a new circle; draw also beyond DE the line *de* parallel to DE, and distant from it by a quantity equal to *ca;* then, by the preceding problem, describe a circle through *c*, which shall touch the circle having for its radius co, and also the straight line *de;* let the centre of this circle be B; if its radius be diminished by the quantity AO or *ca*, the circle described with this new radius will evidently be a tangent to the two given circles, as well as to the straight line DE.

PROBLEM XXVII.

Of inscribing regular polygons in the circle.

The following general method of inscribing regular polygons in the circle is given in various books of practical geometry. On the diameter AB (fig. 40 pl. 5) of the given circle, describe an equilateral triangle; and divide this diameter into as many equal parts as the required polygon is intended to have sides; then from E, the summit of the triangle, draw through *c*, the extremity of the second division, the line E*c*; and continue it till it meet the circumference of the circle in D: the chord AD, they say, will be the side of the required polygon to be inscribed.

We have noticed this method merely to say that it is erroneous, and could be invented only by a person ignorant of geometry, or else intended only as near the truth. For it may be easily demonstrated that it is false, even when employed for finding the simplest polygons, such, for example, as the octagon. It will be found indeed, by trigonometrical calculation, that the angle DCA, which ought to be 45°, is 48° 14′; whence it follows, that the chord AD is not the side of the inscribed octagon.

None of the regular polygons can be inscribed geometrically and without trial, by means of a rule and compasses, except the triangle, and those polygons deduced from it, by doubling the number of sides, as the hexagon, the dodecagon, &c.

The square and those polygons deduced from it in like manner, as the octagon, the sedecagon, &c.

The pentagon and those deduced from it, as the decagon, and the eikosiagon, &c.

The pentedecagon and its derivatives, as the polygon of thirty sides, &c.

The rest, such as the heptagon, enneagon, endecagon, &c, cannot be described by means of the rule and compasses alone, without trial; and all those who have attempted this method have failed, or have produced ridiculous paralogisms.

The following, in a few words, is the method of describing geometrically in a circle, the five primitive polygons, which may be inscribed with the rule and compasses.

Divide the circle ADBE (fig. 41 pl. 5), into four equal parts, by the two diameters AB and DE, intersecting each other at right angles; then divide the radius CD into two equal parts in F, and draw OFG parallel to AB: the line EG will be the side of the inscribed triangle, as well as GO and OE.

The line EB, as every one knows, will be the side of the square.

If EH be made equal to the radius, it is in like manner evident that it will be the side of the hexagon.

Divide the radius AC into two equal parts in I, and draw EI; make IK equal to IC, and the chord EL equal to the remainder EK: EL will be the side of the decagon; and by making the arc LM equal to the arc EL, we shall have the chord EM for the side of the pentagon.

Then divide the arc OM, which is the difference between the arc of the pentagon and that of the triangle, into two equal parts in N, and draw the straight line ON, which will be the side of the pentadecagon, or polygon of 15 sides.

REMARK.

The heptagon is susceptible of a construction, not geometrical, but approximated, which is pretty near the truth, and which on that account deserves to be known; it is as follows: First describe an equilateral triangle, or at least determine the side of one, the half of which will be the side nearly of the insusceptible heptagon. It will be found indeed by calculation, that the side of the triangle, radius being unity, will be equal to 0·86602; the half of which is 0·43301, and the side of the heptagon is 0·43387; the difference therefore between it and half the side of the triangle is less than a thousandth part. Whenever then the thousandth part of the radius of the given circle is an insensible quantity, the above construction will approach very near to the truth.

It is much to be wished that methods of construction equally simple, and as near the truth, could be discovered for all other polygons; which indeed is not possible.

PROBLEM XXVIII.

The side of a polygon of a given number of sides being known; to find the centre of the circumscriptible circle.

This problem is, in some measure, the reverse of the former, and may be easily solved for the same polygons.

We shall say nothing of the triangle, the square and the hexagon, because those who are acquainted with the first elements of geometry know how to find the centre of an equilateral triangle and a square, and that the side of the hexagon is equal to the radius of the circumscriptible circle.

We shall begin therefore with the *pentagon*. Let AB (fig. 42 pl. 5) be the side of the required pentagon; at the extremity of which raise the perpendicular AC, equal to ½ AB; draw BC, and cut off from it CE = AC, and make BF = BE; then with the centre A, and the radius AF, describe an arc of a circle, and from the point B, with the radius BA, describe another intersecting the former in G: the line BG will be the position of the second side of the pentagon, and the two perpendiculars on the middle of the sides AB and BG will give, by their intersection, the position of the centre H.

For the octagon. Let AB (fig. 43 pl. 6) be the given side; on this line describe a semicircle, and raise the radius CG perpendicular, and indefinitely continued; draw the side of the square BG, and make CF equal to the half of BG; draw FE perpendicular to the diameter, and through the point E, where it cuts the semicircle, draw AE, which will meet CG continued in D: this point D will be the centre of the circle required.

For the decagon. If AB (fig. 42 pl. 5) be the given side, find, as if a pentagon were to be constructed, the line BF, and from the points A and B with the radius AF, describe the isosceles triangle A*h*B: the point *h* will be the centre of the decagon.

For the dodecagon, and any other polygons whatever. Let the line given for the side of the polygon be AB (fig. 44 pl. 6). With any radius whatever CD describe a circle, and inscribe in it the required dodecagon or polygon, the side of which we shall suppose to be DE: continue DE to F, if AB exceeds DE, so that DF shall be equal to AB, and then draw CE, and its parallel FG: the point where the latter meets the diameter DH continued, will evidently be the centre of the circle, in which the required polygon is inscriptible.

Though we have given particular methods for the pentagon, octagon and decagon, it is evident that the last method may be applied equally to them all.

We shall conclude this article, on polygons, with two useful tables, one of which contains the sides of the polygons, the radius of the circle being given, and the other the length of the radius, the side of the polygon being known. If the radius of the circle then be expressed by 100000, the side of the inscribed triangle will be within an unit of 173205

that of the square	141421
that of the pentagon	117557
that of the hexagon	100000

that of the heptagon 86777
that of the octagon 76536
that of the enneagon 68404
that of the decagon 61803
that of the endecagon 56347
that of the dodecagon 51763
that of the tredecagon 47844
that of the tesseradecagon . . . 44503
that of the quindecagon 41582

On the other hand, if the side of the polygon be 100000, the radius of the circle will be, in the case

of the triangle 57735
of the square 70710
of the pentagon 85065
of the hexagon 100000
of the heptagon 115237
of the octagon 130657
of the enneagon 146190
of the decagon 161804
of the endecagon 177470
of the dodecagon 193188
of the tredecagon 209012
of the tesseradecagon 224703
of the quindecagon 240488

PROBLEM XXIX.

Method of forming the different regular bodies.

It was long ago demonstrated in geometry that there can be only five bodies, terminated by regular figures, all equal to each other, and forming with one another equal angles. These bodies are:

The tetraedron, which is formed by four equilateral triangles.

The cube, or hexaedron, formed of six equal squares.

The octaedron, formed of eight equal equilateral triangles.

The dodecaedron, formed of twelve equal pentagons.

The icosaedron, formed of twenty equilateral triangles.

Two methods may be employed to form any one of these regular bodies. The first is, to construct a sphere, and then to cut off the excess, so that the remainder shall form the regular body required; the other, which resembles the process used in stone-cutting, consists in first tracing out on a plane, made at hazard, one of the faces of the body to be formed, and then cutting out the adjacent faces, under the determinate angles.

To resolve then the problem in question, we shall first answer the following questions.

1st. The diameter of a sphere being given, to find the sides of the faces of each of the regular bodies.

2d. To find the diameters of the less circles of that sphere, in which the faces of each of these bodies are inscriptible.

3d. To determine the opening of the compasses, with which each of these circles may be described on the surface of the same sphere.

4th. To determine the angles which the contiguous faces form with each other, in their common intersection.

1st. *A sphere being given; to find the sides of the faces of each of the five regular bodies.*

Let ABC (fig. 45 pl. 6) be the half of a great circle of the given sphere, and AC one of its diameters. Divide AC into three equal parts, and let AI be two thirds; draw EI perpendicular to the diameter, cutting the circle in E and join AE: this line will be one of the faces of the tetraedron, and CE will be that of the cube or hexaedron.

Then, through the centre F, draw the radius FB, perpendicular to AC, cutting the circle in B, and join AB: this line AB will be the side of the octaedron inscribed in the same sphere.

The side of the dodecaedron will be found, by dividing

EC, the side of the hexaedron, in mean and extreme ratio, and taking for the side of the dodecaedron the larger segment CK.

Lastly, from A, the extremity of the diameter, draw the perpendicular AG, equal to AC, and from the centre F draw the line FG, intersecting the circle in H: AH will be the side of the icosaedron.

The radius of the circle being 10000, the side of the tetraedron will be found, by calculation, to be equal to 16329; that of the hexaedron or cube, 11546; that of the octaedron, 14142; that of the dodecaedron, 77136; and that of the icosaedron, 10514.

2d. *To find the radius of the lesser circle of the sphere, in which the face of the proposed regular body is inscriptible.*

The method of determining the radius of the circle circumscriptible to the triangle, the square, and the pentagon, which are the only faces of the regular bodies, has been shown already, and consequently the problem is thus solved.

To express them in numbers, as we know that when the side of the equilateral triangle is 10000, the radius of the circumscribing circle is 5773, therefore, as the side of the tetraedron is 16329, nothing is necessary but to say, As 10000 is to 5773, so is 16329 to a fourth proportional, which will be 9426.

It will be found, in like manner, that the radius of the lesser circle, in which the face of the octaedron can be inscribed, is 8164.

And it will be found also, that the radius of the circle in which the face of the icosaedron can be inscribed, is 6070.

The side of the square being 10000, the radius of the circumscribing circle, as is well known, is 7071; which will give for the radius of the face of the hexaedron, 8164.

Lastly, the side of the pentagon being 10000, we shall have for the radius of the circumscribing circle 8506, which will give for the radius of the face of the dodecaedron, 6070.

3d. *To determine the opening of the compasses, with which the circle, capable of receiving the face of the regular body, ought to be described on the sphere.*

This is very easy; for if EF (fig. 46 pl. 6) be the radius of the lesser circle of the sphere, capable of receiving the given face, it is evident that ED is the opening of the compasses proper for describing this circle on the surface of the sphere. But EF is the sine of the angle FCD, which will consequently be given; and FD is the double of the sine of half this first angle: FD therefore may be found by seeking in the tables for the angle FCD, then halving it, afterwards seeking for the sine of that half, and then doubling this sine. This operation will give the value of FD, which in the case of the tetraedron will be 11742; in those of the hexaedron and octaedron, 9192; and in those of the dodecaedron and icosaedron, 6408.

4th. *To find the angle formed by the faces of the different regular bodies.*

Describe a circle (fig. 47 pl. 6) as large as possible, and determine in it the side of the regular body required; if a perpendicular be then let fall from the centre on this side, it will be the diameter of a second circle, which must also be described. We shall here suppose that this diameter is AB.

Describe then, on the side of the regular body found, the proper polygon, or at least find the centre of the circumscribing circle, and from this centre let fall a perpendicular on the side which has been found; in the second circle already mentioned make the lines AD and AC equal to this perpendicular, and the angle DAC will be equal to the angle required.

It will be found, by calculation, that this angle, for the tetraedron, is 70° 32′; for the hexaedron, 90°; this is evident because the faces of the cube are perpendicular to each other; for the octaedron, 109° 28′; for the dodecaedron, 116° 34′; and for the icosaedron, 138° 12′.

We shall here collect all these dimensions in the following table, where we suppose the radius of the sphere to be 10000 parts.

Names of the regular bodies.	Sides of the faces.	Radii of the circumscribing circles.	Distances from the poles.	Angles of the contiguous faces.
Tetraedron	16329	9426	11742	70° 32′
Hexaedron	11546	8164	9192	90 00
Octaedron	14142	8164	9192	109 28
Dodecaedron	77336	6070	6408	116 34
Icosaedron	10514	6070	6408	138 10

It will now be easy to trace out, by either of the above methods, any required regular body whatever.

First method. Let it be required, for example, to form a dodecaedron from a sphere. First describe a circle of a diameter equal to that of the sphere, and determine in it the side of the dodecaedron, or the side of the pentagon, which is one of its faces; also the radius of the circle in which this pentagon can be inscribed, and the opening of the compasses proper for describing it on the sphere; which may be easily done by the precepts before given.

Or, if we suppose the radius of the proposed sphere to be 10000 parts, take upon a scale 6408 of these parts, and with this opening of the compasses describe, on the surface of the sphere, a circle on the circumference of which the five angles of the inscriptible pentagon may be determined; from two neighbouring points describe, with the same opening of the compasses, two arcs, the intersection of which will be the pole of a new circle, equal to the former; continue in this manner, from every two points, and you will have the five poles of the five faces, which rest on the first. In like manner, you may easily determine the other poles, the last of which, if the operation be exact, ought to be diametrically opposite to the first. Lastly, from these twelve poles, describe two equal circles, which will both be cut into five equal parts, and these will determine twelve segments of a sphere, which being cut off, will give the twelve faces of the dodecaedron required.

Second method. Having first found out, on the proposed block, a plane face, describe on it the polygon belonging to the regular body required; then cut out, on each side of this polygon, a new plane, inclined according to the proper angle, as determined in the above table, or which has been traced out by means of the geometrical construction before given; and you will thus obtain so many plane faces, on which new polygons, having one side common with the first polygons, must be described. If the same thing be done on these polygons, you will at length arrive at the last, which, if the operation has been exactly performed, must be perfectly equal to the first.

We must however observe, that perfect exactness will be attained with much more certainty by the first method.

5th. *To form the same bodies of a piece of card.*

If you are desirous of forming these bodies of a piece of card or stiff paper, the following method will be the most convenient.

First trace out on the card all the faces of the required body, viz four triangles for the tetraedron, as seen fig. 48 pl. 6, six squares for the cube, as fig. 49, eight equilateral triangles for the octaedron, fig. 50, twelve pentagons for the dodecaedron, fig. 51 pl. 7, and twenty equilateral triangles for the icosaedron, fig. 52. If you then cut the edges, it will be easy to fold up the faces so as to join, and if they be then glued together, you will have the regular body complete.

The ancient geometricians made a great many geometrical speculations respecting these bodies; and they form almost the whole subject of the last books of Euclid's Elements. A modern commentator on Euclid, M. de Foix Candalle, has even extended those speculations, by inscribing these bodies within each other, and comparing them under different points of view; but, at present, such researches are considered as entirely useless. They were suggested to the ancients by their believing that these bodies were endowed with mysterious properties, on which the explanation of the most secret phenomena of nature depended. With these bodies they compared the celestial orbs, &c. But since rational philosophy has begun to prevail among mankind, the pretended energy of numbers, and that of the regular bodies in nature, have been consigned to oblivion, along with the other visions which were in vogue during the infancy of philosophy, and the reign of platonism. For this reason we shall say nothing farther of these speculations, and confine ourselves to a very curious problem respecting the cube or hexaedron.

PROBLEM XXX.

To cut a hole in a cube, through which another cube of the same size shall be able to pass.

If we conceive a cube raised on one of its angles, in such a manner, that the diagonal passing through that angle shall be perpendicular to the plane which it touches; and if we suppose a perpendicular let fall on that plane from each of the elevated angles, the projection thence resulting will be a regular hexagon, each side and each radius of which may be found in the following manner.

On the vertical line AB (fig. 53 pl. 7), equal to the diagonal of the cube, or the square of which is triple to that of the cube, describe a semicircle, and make AC equal to the side of the cube, and AD equal to the diagonal of one of its faces; if from the point C there be let fall, on the horizontal tangent of the circle in B, the perpendicular CE, passing through the point D, BE will be the side and the radius of the required hexagon *abcd* fig. 54.

When this operation is finished, describe on this hexagonal projection, and around the same centre, the square which forms the projection of the given cube placed on one of its bases, so that one of its sides shall be parallel, and the other perpendicular to the diameter *ac*: it may be demonstrated, that this square can be contained within the hexagon, in such a manner, as not to touch with its angles any of the sides: a square hole therefore, equal to one of the bases of the cube, may be made in it, in a direction parallel to one of its diagonals, without destroying the continuity of any side; and consequently another cube of equal size may pass through it, provided it be made to move in the direction of the diagonal of the former.

PROBLEM XXXI.

With one sweep of the compasses, and without altering the opening, or changing the centre, to describe an oval.

This problem, as is the case with others of a similar kind, is a mere deception; for it is not specified on what kind of surface the required curve ought to be described. Those to whom this problem is proposed will think of a plane surface, and therefore will consider it impossible, as it really is; while indeed the surface meant is a curved one, on which it may be easily performed.

If a sheet of paper be spread round on a cylindric surface, and if a circle be described upon it with a pair of compasses, assuming any point whatever as a centre, it is evident that, when the sheet of paper is extended on a plane surface, we shall have an oval figure, the shortest diameter of which will be in the direction corresponding to that of the axis of the cylinder.

We should however be deceived were we to take this curve for the real ellipsis, so well known to geometricians. The method of describing the latter is as follows.

PROBLEM XXXII.

To describe a true oval or ellipsis geometrically.

The geometrical oval is a curve with two unequal axes, and having in its greater axis two points so situated, that if lines be drawn to these two points, from each point of the circumference, the sum of these two lines will be always the same.

Let AB (fig. 55 pl. 7) then be the greater axis of the ellipsis to be described; and let DE, intersecting it at right angles, and dividing it into two equal parts, be the lesser axis, which is also divided into two equal parts in C; from the point D as a centre, with a radius equal to AC, describe an arc of a circle, cutting the greater axis in F and *f*: these two points are what are called the foci: fix in each of these a pin, or, if you operate on the ground, a very straight peg; then take a thread, or a chord if you mean to describe the figure on the ground, having its two ends tied together, and in length equal to the line AB, plus the distance F*f*; place it round the pins or pegs F, *f*; then stretch it as seen at FG*f*, and with a pencil, or sharp pointed instrument, make it move round from B, through D, A, and E, till it return again to B: the curve described by the pencil on paper, or on the ground by any sharp instrument, during a whole revolution, will be the curve required.

This ellipsis is called the Gardener's Oval; because when gardeners describe that figure they employ this method.

It is here seen that the geometric ellipsis, or oval, is, as we may say, a circle with two centres; for in the circle the distance from the centre to any point of the circumference, and from that point back to the centre, is always equal to the same sum, viz the diameter. In the ellipsis, where there are two centres, the distance from one of them to any point of the circumference, and from that point to the other centre, is always equal to the same sum, or to the greater diameter.

A circle therefore is nothing else than an ellipsis, the two foci of which, by continually approaching, have at length been united and confounded with each other.

Another method of describing an ellipsis, which may be also used some times, is as follows.

Let ABC (fig. 6 pl. 7) be a square, and BH and BI the two semi-axes of the ellipsis to be described. Provide a rule, such as ED, equal to the sum of these two lines, and having taken EF equal to BH, fix in the point F, by some mechanism which may be easily invented, a pencil or piece of chalk, capable of tracing out a line upon paper; then make this rule turn in the given right angle, in such a manner, that its two extremities shall always touch the sides of that angle, and during this movement, the pencil fixed in F will describe a real geometrical ellipsis.

It may be readily seen, that if the pencil or chalk were fixed in the point G, which divides DE into two equal parts, the curve described would be a circle.

REMARK.

Another sort of oval, very much used by architects and engineers, when they intend to form a flat or an acute arch, is called by the French workmen *anses de paniers.* It consists of several arcs of circles having different radii, which mutually touch each other, and which represent pretty nearly a geometrical ellipsis. But it has one fault, which is, that however well these arcs touch each other, a nice eye will always observe at the place of junction an inequality, which is the effect of the sudden transition of one curve to another that is larger. For this reason, any arch which rises on its pier without an impost seems to form an inequality, though the arch at its junction with the pier may touch it exactly.

This inconvenience however is compensated by one advantage, which is, that for the *voussoirs* of the arch, there is no need but of two *panneaux*, or model boards, if the quarter of the oval be formed of two arcs, or of three if it be formed of three; whereas, if it were a real ellipsis, it would have occasion for as many *panneaux* as *voussoirs*. If any one however should have the courage, and it would require no small degree of it, to surmount this difficulty, we entertain no doubt that the real ellipsis would have more beauty than this bastard kind of it.

PROBLEM XXXIII.

On a given base to describe an infinite number of triangles, in which the sum of the two sides, standing on the base, shall be always the same.

This is only a corollary to the preceding problem. For on a given base let there be described an ellipsis, having the two extremities of that base at its foci: all the points of the ellipsis will be the summits of as many triangles on the given base FG*f*, F*gf*, (fig. 55 pl. 7), and the sum of their sides will be the same; consequently they will all have the same perimeter, and the greatest triangle will be that which has its two sides equal; for it is that which has the summit at the most elevated part of the ellipsis.

THEOREM VI.

Of all the isoperimetric figures, or figures having the same perimeter, and a determinate number of sides, the greatest is that which has all its sides and all its angles equal.

We shall first demonstrate this theorem in regard to triangles. Let ACB (fig. 57 pl. 7) then be a triangle on the base AB, the sides of which AC and CB are unequal. We have already shown, that if there be constructed a triangle AFB, the equal sides of which AF and FB are together equal to AC and CB, the triangle AFB will be greater than ACB.

For the same reason, if there be constructed, on AF as a base, the triangle A*b*F, the sides of which A*b* and *b*F are equal to each other, and together equal to AB and BF, the triangle A*b*F will be greater than AFB. In like manner, if we suppose F*a* and *ab* equal, and their sum equal to FA and AB, the latter triangle F*ab* will be still greater than AFB, which has the same perimeter, &c. But it may be readily seen by this operation, that the three sides of the triangle always approximate towards equality, and that, by conceiving it continued ad infinitum, the triangle would at length become equilateral, and consequently the equilateral triangle will be the greatest of all.

For example, if the three sides of the first triangle be 12, 13, 5, the sides of the second will be 12, 9, 9; those of the third 9, $10\frac{1}{2}$, $10\frac{1}{2}$; those of the fourth $10\frac{1}{2}$, $9\frac{3}{4}$, $9\frac{3}{4}$; those of the fifth $9\frac{3}{4}$, $10\frac{1}{8}$, $10\frac{1}{8}$; those of the sixth $10\frac{1}{8}$, $9\frac{15}{16}$, $9\frac{15}{16}$; those of the seventh $9\frac{15}{16}$, $10\frac{1}{32}$, $10\frac{1}{32}$; and so on; by which it is seen that the difference always decreases; so that at last the three sides become 10, 10, 10, and the triangle will then be the greatest of all.

If we now take a rectilineal polygon, such as ABCDEF (fig. 58 pl. 7), all the sides of which are unequal: draw the lines AC, CE, and EA. By what has been already shown it will be seen, that if an isosceles triangle A*b*C be described on AC, in such a manner, that A*b* and *b*C shall be together equal to AB and BC, the polygon, though of the same perimeter, will become greater by the excess of the triangle A*b*C above ABC. If the same thing be done all around, the surface of the polygon will be continually augmented; all its sides and its angles will more and more approach to equality, and consequently the greatest of all will be that which has all its sides and angles equal.

We shall now demonstrate, that, of two regular polygons, having the same perimeter, the greater is that which has the greatest number of sides. For this purpose, let any polygon, an equilateral triangle for example, be circumscribed round a circle, and let KFHI (fig. 59 pl. 7) be an hexagon circumscribed about the same circle: it is evident that the perimeter of the latter will be less than that of the triangle; for the parts FE, GH, and IK, are common, and the side GF is less than FB plus BG, &c; a hexagon, concentric to the former, and equal in perimeter to the triangle, which we here suppose to be MNO, will therefore be without the hexagon KFH; consequently the perpendicular *cl* will be greater than CL. But as the triangle has the same perimeter as the hexagon MNO, their areas will be as the perpendiculars CL, *cl*, let fall from the centre of the circle; and therefore the hexagon, having the same perimeter as the triangle, will be the greater.

What has been demonstrated in regard to a triangle and hexagon of the same perimeter, is evidently applicable to any other two polygons, one of which has a number of sides double to that of the other; consequently the more sides a polygon of a determinate perimeter has, the greater is its area.

REMARKS.

1st. This leads us to a consequence much celebrated in geometry, which is: that of all the figures, having the same perimeter, the circle is of the greatest capacity; for a circle is only a polygon of an infinite number of sides, or, to use a more geometrical expression, is the last of the polygons resulting from their sides being continually doubled: consequently it is the greatest of all.

2d. We shall here remark also, that if upon any determinate base, and with a determinate perimeter, there be described several figures, the greatest will be that which has the greatest number of sides, besides the base, and which approaches nearest to regularity; hence it follows, that if it be required to describe, with a determinate length, on a given base, the greatest figure, that figure will be the segment of a circle, viz, a segment having that base for its chord, and for its arc the given length.

All these things may be demonstrated by a mechanical consideration. For let us suppose a vessel, the sides of which are flexible, and that any liquor is poured into it; the sides it is certain will arrange themselves in such a manner as to contain the greatest quantity possible. On the other hand, it is well known that the vessel will assume the cylindric form; that is to say, its base and the sections parallel to the base will be circular; hence it follows that, of all figures having the same perimeter, the circle is that which comprehends the greatest area.

By means of the above observations it will be easy to solve the following questions.

I.

A *has a field* 500 *poles in circumference, which is square;* B *has one of the same circumference which is an oblong, and proposes to* A *an exchange. Ought the latter to accept the offer?*

It is easy to answer that he ought not; and A would sustain more loss by the exchange the greater the inequality is between the sides of the field belonging to B. This inequality might even be such, that the latter field would be only the half, or the fourth, or the tenth part of that of A. For let us suppose the field of A to be 100 poles on each side; and that the field of B is a rectangle, one side of which is 190 poles, and the other 10, by which means it will have the same perimeter as the other; it will however be found that the surface of the latter will be only 1900 square poles, while that of the former will be

10000. If one side of the field belonging to B were 195 poles, and the other 5, which would still make the perimeter 400 poles; its surface would be only 975 poles, which is not even a tenth part of that of the field belonging to A.

II.

A farmer borrowed a sack of wheat, measuring 4 feet in length, and 6 feet in circumference; for which he returned two sacks, of the same length, and each 3 feet in circumference: did he return the same quantity of wheat?

He returned only half the quantity; for two equal circles, having the same perimeter, taken together, as a third, do not contain the same area; the area of both is only the half of the third, each of them being but a fourth of it.

III.

A green-grocer purchased for a certain sum, as many heads of asparagus, as could be contained in a string a foot in length; being desirous to purchase double that quantity, he returned next day to the market, with a string of twice the length, and offered double the price of the former quantity, for as many as it would contain. Was his offer reasonable?

No—the man was in an error to imagine that a string of twice the length would contain only double the quantity of what he purchased the preceding day; for a circle which has its circumference double to that of another, has its diameter double also. But the area of a circle, the diameter of which is double to that of another, is equal to four times the area of the other.

REMARK.

It remains for us to observe here, that as the circle, of all the figures having an equal perimeter, is the greatest; the sphere among the solids, is that which contains the greatest volume. Thus, if it were required to make a vessel of a determinate capacity, but in such a manner as to save the materials as much as possible, it ought to be in the form of a sphere. But this will be better illustrated by the following problem.

PROBLEM XXXIV.

A gentleman wishes to have a silver vessel of a cylindric form, open at the top, capable of containing a cubic foot of liquor; but, being desirous to save the material as much as possible, requests to know the proper dimensions of the vessel.

If we suppose that the vessel ought to be a line in thickness, for example, it is evident that the quantity of the matter will be proportional to the surface. The question then is: Of all the cylinders, capable of containing a cubic foot, to determine that which shall have the least surface, exclusive of the top.

It will be found that the diameter of the base ought to be 16 inches 4 lines; and the height 8 inches $2\frac{2}{10}$ lines, which is the ratio of nearly 2 to 1 between the diameter and the height.

If it were required to have the vessel in the form of a cask, close at both ends, the question would be: To find a cylinder which shall have its whole surface, comprehending the two bases, greater than that of any other of the same capacity. In this case the diameter of the base ought to be 13 inches, and the height 12 inches $5\frac{1}{4}$ lines.

PROBLEM XXXV.

On the form in which the bees construct their combs.

The ancients admired bees on account of the hexagonal form of their combs. They observed that, of all the regular figures which can be united, without leaving any vacuum, the hexagon approaches nearest to the circle, and with the same capacity has the least perimeter; whence they inferred that this animal was endowed with a sort of instinct, which made it choose this figure as that, which containing the same quantity of honey, would require the least wax to construct the comb; for it appears that bees do not prepare wax on its own account, but in order to construct their combs destined to be the repositories of their honey, and receptacles for their young.

This however is far from being the principal wonder in regard to the labour of bees, if we can give the name of wonder to a labour blindly determined by a peculiar organization; for it may be remarked, in the first place, that it is not absolutely wonderful that small animals, all endowed with the same activity and the same force, pressing outwards, from within, small cells all arranged close to each other, and all equally flexible, should give them, by a sort of mechanical necessity, a hexagonal form. If we suppose indeed a multitude of circles, or small cylinders, highly flexible and somewhat extensible, close to each other, and that forces acting internally, and all equal, tend to make their sides approach each other, by filling up the vacuities left between them, the first form they will assume will be the hexagonal; after which all these forces remaining in equilibrium, nothing will tend to change that form.

However, not to deprive the bees of the admiration which they have excited in the above respect, we shall remark that this is not the manner in which they labour. They do not first make circular cells, and then transform them into hexagons by extending them in concert. The cells, which terminate an imperfect comb, are composed of equal planes inclined to each other, nearly in that angle which the hexagonal form requires. But let us proceed to another singularity, still more wonderful, in regard to the labour of bees.

This singularity consists in the manner in which the bottom of their cells is formed. We must not indeed imagine that they are all uniformly terminated by a plane perpendicular to their axes; there is a method of terminating them which employs less wax, and even the least possible, still leaving to the cells the same capacity; and it is this method which these insects adopt, and which they execute with great precision.

To execute this disposition, it is necessary, in the first place, that the two rows of cells, of which it is well known a comb consists, and which stand back to back, should not be arranged so as to make their axes correspond, but in such a manner that the axis of the one may be in a line with the common juncture of the three posterior. As is seen fig. 60 pl. 7, where the hexagon described with black lines corresponds with the three formed of dotted lines, which represent the plane of the posterior cells; and it is thus that the cells of bees are arranged, to suit the disposition of their common bottoms.

In the second place, to give an idea of this disposition, let us suppose an hexagonal prism, the upper base of which is the hexagon ABCDEF (fig. 61 pl. 8), with a triangle AEC inscribed in it. Let the axis FO be continued to S, and through the point S and the side AC let a plane pass, which shall cut off from the prism the angle B, so as to form a rhomboidal face ASCT: such is one of the bottoms of the cell of a comb: if two other similar planes be made to pass through S and the sides AE and EC, they will form the other two; so that the bottom is terminated by a triangular pyramid.

It may be readily seen, that wherever the point S may be situated, as the pyramid ACOS is always equal to ACBT, and as the case is the same with the rest, the capacity of the cell will not vary, whatever be the inclination of that part of the bottom turned towards AC. But the case is different with the surface where there is such an inclination, that the whole surface of the prism and of its bottom will be less than with any other inclination. It has been found by the researches of geometricians, that, for this purpose, the angle formed by the bottom with the axis ought to be 54° 44′; from which there results the smaller angle of the rhombus ATC or ASC, equal to 70° 32′, and the other SAT or SCT of 109° 28′.

But this is exactly the inclination of the sides of the parallelogram, formed by each of the three inclined planes of the bottom of the cells of a comb, as appears by the measurement of a great many of these cells. Hence there is reason to conclude, that bees construct the bottom of their cells in the most advantageous form, so as to have the least surface possible, and in such a manner indeed, as can be determined only by the modern geometry *. Who can have given to these insects, so contemptible, not in the eyes of the philosopher, who never despises the least of the works of the Deity, but in the eyes of the vulgar, that wonderful instinct, which directs them to perform so perfect a work, but the supreme Geometrician, of whom Plato said, what is verified more and more as we become acquainted with the works of nature, that he does every thing *numero, pondere et mensura.*

PROBLEM XXXVI.

What is the greatest polygon that can be formed of given lines?

It may be demonstrated that the greatest polygon that can be formed with given lines, is that about which a circle can be circumscribed.

But it may be still asked, whether there be any particular order, in regard to the sides, capable of giving a greater polygon than any other arrangement. We can answer,

* The Abbé Delisle says improperly, in the notes to the fourth book of his Translation of the Georgics, that Reaumer having proposed this problem to Kœnig, the latter, *after a great many calculations*, at length found the angle of the inclination of the planes which form the bottom of these cells. Nothing however is easier than the solution of this problem by means of fluxions: two lines of calculation are sufficient, and a solution may even be given without that assistance.

that there is not; and that whatever be the arrangement, if the polygon can be inscribed in a circle, it will be always the same; for it may be easily demonstrated, that whatever be this order, the size of the circle will not vary; the polygon will always be composed of the same triangles, having their summits at its centre: the only difference will be, that they will be differently arranged.

PROBLEM XXXVII.

What is the largest triangle that can be inscribed in a circle; and what is the least that can be circumscribed about it?

The triangle required in both these cases is the equilateral.

The case is the same with the other polygons. The greatest quadrilateral figure that can be inscribed in the circle, is the square; this figure also is the least of all those that can be circumscribed about a circle.

The regular pentagon is likewise the greatest of all the five-sided figures that can be inscribed in the circle; and the same figure is the least of all the pentagons that can be circumscribed about the circle. And so on.

PROBLEM XXXVIII.

AB (fig 62 pl. 8) *is the line of separation between two plains; one of which* ACIB *consists of soft sand, in which a vigorous horse can scarcely advance at the rate of a league per hour; the other* ABDK *is covered with fine turf, where the same horse, without much fatigue, can proceed at the rate of a league in half an hour; the two places* C *and* D *are given in position, that is to say the distance* CA *and* DB *of each from the line of boundary* AB, *as well as the position and length of* AB, *are known; now if a traveller has to go from* D *to* C, *what route must he pursue, so as to employ the least time possible on his journey?*

Most people, judging of this question according to common ideas, would imagine that the route to be pursued by the traveller ought to be the straight line. In this however they would be deceived, as may be easily shown; for if the straight line CED be drawn, it may be readily conceived that it will be gaining an advantage to perform, in the first plain, where it is difficult to travel, the part of the journey CF, which is somewhat shorter than CE; and to perform in the second, where it is much easier to travel, the part FD, longer than DE, that is to say, than the space which would be passed over by going directly from C to D; so that less time would really be employed to go from C to D, by CF and FD, than by CE and ED, though the road by the latter is shorter.

This indeed may be demonstrated by calculation. For if HG be drawn perpendicular to AB, through the point F, it will be found that one can go from C to D, in the least time possible, when the sines of the angles CFG and DFH, are to each other respectively in the inverse ratio of the velocity with which the traveller can pass over the planes ACIB and ABDK, that is to say, in the present case, as 1 to 2; and therefore the sine of the angle CFG ought to be half only of that of the angle DFH.

PROBLEM XXXIX.

On a given base to describe an infinite number of triangles, in such a manner, that the sum of the squares of the sides shall be constantly the same, and equal to a given square.

Let AB (fig. 63 and 64 pl. 8) be the given base, which must be divided into two equal parts in C; then from the points A and B, with a radius equal to half the diagonal of the given square, describe an isosceles triangle, of which F is the vertex; draw CF, and from the point C, with the radius CF, describe a semicircle on AB, produced if necessary: all the triangles having AB for their base, and whose vertices are at F, f, φ, in the circumference of the circle, will be of such a nature, that the sum of the squares of their sides will be equal to the square given.

REMARK.

Every one knows that when the sum of the squares of the sides is equal to the square of the base, the triangle is right-angled, and has its vertex in the circumference of the circle described on that base. Here it is seen, that if the sum of the squares of the sides is greater or less than the square of the base, the vertices of the triangles, which in this first case are acute-angled, and in the second obtuse-angled, are always in a semicircle also, having the same centre, but on a diameter greater or less than the base of the triangle; which is a very ingenious generalization of the well-known property of the right-angled triangle.

PROBLEM XL.

On a given base, to describe an infinite number of triangles, in such a manner, that the ratio of the two sides, on that base, shall be constantly the same.

Divide the given base AB (fig. 65 pl. 8) in such a manner in D, that AD may be to DB, in the given ratio, which we shall here suppose to be as 2 to 1. Then say, as the difference between AD and DB is to DB, so is AB to BE; and if AD exceeds DB, BE must be taken in the direction ABE; then divide DE into two equal parts in C, and from C as a centre, with the radius CD or CE, describe a semicircle on the diameter DE: all the triangles, as AFB, A*f*B, AϕB, &c, having the same base AB, and their vertices F, *f*, ϕ, in the circumference of this semicircle, will be of such a nature, that their sides AF, FB; A*f*, *f*B; Aϕ, ϕB, will be in the same ratio, viz, that of AD to DB, or of AE to EB, which is the same thing.

But the centre C will be found much easier by the following construction; on AD describe the equilateral triangle AGD, and on DB the equilateral triangle DHB; through their summits, G and H, draw a straight line, which being continued will cut the continuation of AB in the point C, and this point will be the centre required.

THEOREM VII.

In a circle, if two chords, as AB *and* CD (fig. 66 pl. 8), *intersect each other at right angles; the sum of the squares of their segments,* CE, AE, ED, *and* EB, *will always be equal to the square of the diameter.*

The demonstration of this curious and elegant theorem is exceedingly easy; for it may be readily seen, if the lines BD and AC be drawn, that their two squares are together equal to the squares of the four segments in question. Moreover, by making the arc FC equal to AD, we shall have the arc FD equal to AC, and consequently the angle FDC equal to ACE, which is itself equal to ABD; the angle FDB therefore will be a right angle, since it is equal to EDB and DBE, which together make a right angle; hence the squares of FD and DB are equal to the square of the hypothenuse FB, which is the diameter.

It must here be remarked, that the result would be the same, if we suppose the point *e*, where the chords meet, to be without the circle; in that case the four squares, viz, those of *ea*, *eb*, *ec* and *ed* would still be together equal to the square of the diameter.

REMARK.

Circles being to each other as the squares of their diameters; it is evident that if on EA, EB, EC, and ED, as diameters, four circles be described, these circles will be together equal to the circle ACBD. And they will also be proportional; for we know that BE is to EC as ED is to EA. But if four magnitudes are proportional, their squares are so also. Moreover, it is evident that whatever be the position of these two chords, their sum will always be equal, at the most, to two diameters if they both pass through the centre, or at least to one, if one of them passes through the centre, and the other almost at the distance of a radius. By means of this theorem therefore, it will be easy to solve the following problem.

PROBLEM XLI.

To find four proportional circles, which taken together shall be equal to a given circle, and whieh shall be of such a nature, that the sum of their diameters shall be equal to a given line.

It is evident, for the above reasons, that the given line must be less than twice the diameter of the given circle, and greater than once that diameter; or, which is the same thing, that the half of this line must be less than the diameter of the given circle, and greater than its radius.

This being premised; let the given line, or the sum of the diameters of the required circles, be *ab* (fig. 67 pl. 8), the half of which is *ac*; let ADBE be the given circle, the two diameters of which are AB and DE, perpendicular to each other. On the radii CA and CE continued, make the lines CF and CG equal to *ac*, and draw FG, which will necessarily intersect CH, the square of the radius of the circle. In the part IK of that line comprehended within the square, assume any point L, from which draw the lines LM*q*, and LN*r*, the one parallel and the other perpendicular to the diameter AB; through the points M and N, where they intersect the circumference of the circle, draw MR and NQ, the one parallel and the other perpendicular to AB: the chords NS and MT will be the two chords required.

For it is evident that NQ and MR are equal to L*q* and L*r*, which are together equal to CG or CF, or to the half of *ab*; the whole chords then are together equal to *ab;* consequently, by the preceding theorem, they solve the problem, and the four circles described on the diameters NO, OM, OS, and OT, will be equal to the circle ADBE.

REMARK.

The line FG may happen only to touch the circle; in which case any point, except the point of contact, will equally solve the problem.

But if FG intersect the circle, as seen (fig. 68), the point L must be assumed in that part of the line IK, which is without the circle, as seen in the same figure.

This solution is much better than that given by M. Ozanam; for he tells us to take on *ac* (fig. 67) a portion less than the radius, and to set it off from C to *q*; then to draw the lines *q*M and MR, and to set off the remainder of *ac* from C to *r*; but it is necessary that the point *r* should fall beyond R, otherwise the two semi-chords would not intersect each other. In the last place, according to the magnitude of *ac*, in regard to the radius, there is a certain magnitude which must not be exceeded, and which M. Ozanam does not determine: this therefore renders the solution defective.

PROBLEM XLII.

Of the Trisection and Multisection of an Angle.

This problem is celebrated on account of the fruitless attempts made, from time to time, to resolve it geometrically, by the help of a rule and compasses, and of the paralogisms and false constructions given by pretended geometricians. But it is now demonstrated, that the solution of it depends on a geometry superior to the elementary, and that it cannot be effected by any construction in which a rule and compasses only, or the circle and straight line, are employed, except in a very few cases; such as those where the arc which measures the proposed angle is a whole circle, or a half, a fourth, or a fifth part of one. None therefore but people ignorant of the mathematics attempt at present to solve this problem by the common geometry.

But though it cannot be solved by the rule and compasses alone, without repeated trials, there are some mechanical constructions or methods, which, on account of their simplicity, deserve to be known. They are as follow:

Let it be proposed, for example, to divide the angle ABC (fig. 69 pl. 8) into three equal parts. From the point A let fall, on the other side of the angle, the perpendicular AC, and through the same point A draw the indefinite straight line AE parallel to BC; if from the point B you then draw to AE a line BE, in such a manner, that the part FE, intercepted between the lines AC and AE, shall be equal to twice the line AB, which may be done very easily by repeated trial, you will have the angle FBC equal to the third part of ABC.

If FE indeed be divided into two equal parts in D, and if AD be drawn; as the triangle FAE is right-angled, D will be the centre of the circle passing through the points F, A, E; consequently DA, DE, and DF will be equal to each other, and to the line AB; the triangle ADE then will be isosceles, and the angles DAE and DEA will be equal; the external angle ADF, which is equal to the two interior ones DAE and DEA, will therefore be the double of each of them. But as the triangle BAD is isosceles, the angle ABD is equal to ADB, and the angle AED, or its equal FBC, is half of the angle ABD; consequently the angle ABC is divided by BE, in such a manner, that the angle EBC is the third part of it.

Another method. Let the given angle be ACB (fig. 70 pl. 9): from the vertex of it as a centre, describe a circle, and continue the radius BC indefinitely to E; then draw the line AE in such a manner, that the part DE, intercepted between BE and the circumference of the circle, shall be equal to the radius BC; if CH be then drawn through the centre C, parallel to AE, the angle BCH will be the third part of the given angle ACB.

If the radius CD be drawn, it may be readily seen that the angle HCA is equal, on account of the parallel lines, to CAD or CDA. But the latter is equal to the angles DCE and DEC, or to the double of one of them; since CD and DE are equal by construction; and as the angle HCB is equal to DCE or DEC, the angle ACH is the double of HCB, and consequently ACB is the triple of HCB.

PROBLEM XLIII.

The Duplication of the Cube.

To double a rectilineal surface, or any curve whatever, as the circle, square, triangle, &c, is easy; that is to say, one of these figures being given, it is easy to construct a similar one, which shall be the double or any multiple of it whatever, or which shall be in any given ratio to it at pleasure; nothing is necessary for this purpose, but to find the mean geometrical proportional between one of the sides of the given figure, and the line which is to that side in the given ratio; this mean will be the side homologous to that of the given figure. Thus, to describe a circle double of another, a mean proportional must be found between the diameter of the former and the double of that diameter; this proportional will be the diameter of the double circle, &c. The case is the same with every other ratio.

All this belongs to the elements of geometry. But to construct a double solid figure, or a figure in a given ratio to another similar figure, is a much more difficult problem, which cannot be solved by means of the circle and straight line, or of the rule and compasses, unless a method of repeated trial, which geometry rejects, be employed. This at present is clearly demonstrated; but the demonstration is not susceptible of being comprehended by every one.

Respecting the origin of this problem, a very curious circumstauce is related. During the plague at Athens, which made a dreadful havoc in that city, some persons being sent to Delphos to consult Apollo, the deity promised to put an end to the destructive scourge, when an altar, double to that which had been erected to him, should be constructed. The artists who were immediately dispatched to double the altar, thought they had nothing to do, in order to comply with the demand of the oracle, but to double its dimensions. By these means it was made octuple; but the god, being a better geometrician, wanted it only double. As the plague still continued, the Athenians dispatched new deputies, who received for answer, that the altar was more than double. It was then thought proper to have recourse to the geometricians, who endeavoured to find out a solution of the problem. There is reason to think that the god was satisfied with an approximation, or mechanical solution; had he required more, the situation of the people of Athens would have deserved pity indeed.

There was no necessity for introducing a deity into this business. What is more natural to geometricians than to try to double a solid, and the cube in particular, after having found the method of doubling the square and other surfaces? This is the progress of the human mind in geometry.

Geometricians soon observed that, as the duplication of any surface consists in finding a geometrical mean between two lines, one of which is the double of the other, the duplication of the cube, or of any solid whatever, consists in finding the first of two continued mean proportionals between the same lines. We are indebted for this remark to Hippocrates of Chios, who from being a wine merchant, ruined by shipwreck or the officers of the excise at Athens, became a geometrician. Since that time, all the efforts of geometricians have been confined merely to the finding of two continued geometrical mean proportionals between two given lines, and these two problems, viz, that of the duplication of the cube, or, more generally, of the construction of a cube in a given ratio to another, and that of the two continued mean proportionals, have become synonimous.

The different methods of solving this problem, some of which require repeated trial, and some no other instruments than a rule and compasses, are as follow:

1st. Let the two lines, between which it is required to find two mean proportionals, be AB and AC (fig. 71 pl. 9). Form of them the rectangle BACD, and continue the sides AB and AC indefinitely; draw the two diagonals of the rectangle intersecting each other in E; and we shall then have the solution of the problem, if the line FDG, terminated by the sides of the right angle FAG, be drawn through the point D, in such a manner, that the points G and F shall be equally distant from the point E; for in that case the lines AB, CG, BF, and AC, will be in continued proportion.

Or, with E as a centre, describe an arc of a circle, as FIG, in such a manner, that by drawing the line FG, it shall pass through the point D; we shall then have a solution of the problem.

Another method is as follows: Circumscribe a circle about the rectangle BACD; then through the point D draw the line FG, in such a manner, that the segments FD and GH shall be equal: the lines CG and BF will be continued mean proportionals between AB and AC.

2d. Form a right angle of the two given lines AB and BC (fig. 72 pl. 9); and having continued BC and AB indefinitely, from the point B, as a centre, describe the semicircle DEA; draw also the line AC, and in the continuation of it find a point G of such a nature, that by drawing the line DGHI, the segments GH and HI shall be equal to each other: the line BH will be the first of the two means.

3d. Let CA (fig. 73 pl. 9) be the first of the given lines: from the point C, with the radius CB, equal to the half of CA, describe a circle, and in this circle make the chord BD equal to the second of the given lines, which must be continued indefinitely; draw the indefinite line ADB, and from the point C draw the line CEF, in such a manner, that the part EF, intercepted within the angle EDF, shall be equal to CB; the line DF will then be the first of the required mean proportionals, and CE will be the second. This construction is that of sir Isaac Newton.

PROBLEM XLIV.

An angle, which is not an exact portion of the circumference, being given, to find its value with great accuracy by means of a pair of compasses only.

From the vertex of the given angle, with as great a radius as possible, describe a circle, and mark its principal points of division, as the half, third, fourth, fifth, sixth, eighth, twelfth, and fifteenth parts of the circumference; then by means of the compasses take the chord of the given arc, and set it off along the circumference, from a determinate point, going round it once, twice, thrice, &c; and counting the number of times that the chord is applied to the circumference, until you fall exactly on one of the points of division, which cannot fail to be the case after a certain number of revolutions, unless the given arc be incommensurable to the circumference; then examine what the point of division is, or how many and what aliquot parts of the circumference it is distant from the first point; add the number of degrees which it gives to the product of 360 degrees multiplied by the complete number of turns made with the compasses, and divide the sum by the number of times that the compasses were applied to the circumference: the quotient will be the number of degrees, minutes and seconds required.

Let us suppose for example, that the compasses, with an opening equal to the chord of the given arc, have been applied to the circumference seventeen times, and that after four complete revolutions they have coincided exactly on the second division of the circle divided into five equal parts. The fifth part of the circumference is 72°, and two fifths are 144°; if 144 then be added to the product of 360° by 4, which is the number of the complete revolutions, and if the sum 1584° be divided by 17, the quotient 93° 10′ 35″ will be the value of the required arc.

PROBLEM XLV.

A straight line being given; to find, by an easy operation, and without a scale, to a thousandth, ten thousandth, hundred thousandth, &c part, nearly, its proportion to another.

Let the first or least of these lines be called A, and second B.

Take with a pair of compasses the extent of the line A, and set it off as many times as possible on B: we shall here suppose that A is contained in the latter 3 times, with a remainder.

Take this remainder in the compasses, and set it off, in like manner, on the line B, as often as possible: we shall suppose that it is contained in it 7 times, with a remainder.

Take the second remainder, and perform the same operation on the line B, in which we shall suppose it to be contained 13 times, with a remainder; and, in the last place, let us suppose that this third remainder is contained in B exactly 24 times.

Then form the following series of fractions; $\frac{1}{3}$, $\frac{1}{3.7}$, $\frac{1}{3.7.13}$, $\frac{1}{3.7.13.24}$, and reduce them to decimal fractions, which will be 0·333333, 0·047619, 0·003663, 0·000152. The given line is in decimals equal to the first of these fractions, minus the second, plus the third, minus the fourth, which gives 0·289225, without the error of one of these parts entirely, that is to say of a millionth part.

It may be easily seen that no scale, however small the divisions, could give so approximate a ratio; and even if we suppose such a scale to exist, there would still remain an uncertainty in regard to the division on which the extremity of the given line would fall; whereas, a line applied with the compasses along a greater one, can never leave any uncertainty in regard to the number of times it is contained in it, with or without a remainder.

If the above fractions be added, in the usual manner, we shall find that the given line is equal to $\frac{1895}{6552}$ of the second.

PROBLEM XLVI.

To make the same body pass through a square hole, a round hole, and an elliptical hole.

We give a place to this pretended problem, merely because it is found in all the *Mathematical Recreations* hitherto published; for nothing is easier to those who are in the least acquainted with the simplest geometrical bodies.

Provide a right cylinder, and suppose it to be cut through its axis; this section will be a square or a rectangle; if cut through a plane perpendicular to the axis, the section will be a circle; and if cut obliquely to that axis, the section will be an ellipsis. Consequently, if three holes, the first equal to this rectangle, the second to the circle, and the third to the ellipsis, be cut in a piece of wood or pasteboard, it is evident that the cylinder may be made to pass through the first of these holes, by moving it in a direction perpendicular to its axis; it will also pass through the circular hole when moved in the direction of its axis; and through the elliptical hole, when held with the proper degree of obliquity; in all these cases it will exactly touch the edges of the hole, so that if the hole were smaller it would be impossible to make it pass through it.

This problem might be solved by means of other bodies; but it is so simple that nothing farther needs be said on the subject.

PROBLEM XLVII.

To measure the circle, that is to say, to find a rectilineal space equal to the circle, or more generally, to find a straight line equal to the circumference of the circle, or to a given arc of that circumference.

We are far from pretending to give an exact and perfect solution of this problem: it is more than probable that it will ever baffle the efforts of the human mind; but it is allowed in geometry, that when a problem cannot be completely solved, it is some merit to approach near to it, and the more so when the unknown quantity is circumscribed within the nearest limits. But though geometricians despair of ever being able to find the exact measure of the circle, they have accomplished things highly worthy of notice; for they have found means to approach so near to it, that even if the radius of a circle were equal to the distance between the sun and the first of the fixed stars, it is certain that its circumference might be found from the radius, without the error of a hair's breadth. This is doubtless more than sufficient to answer the nicest purposes in the arts; but it must be allowed that it would give great pleasure to a geometrical genius, to be able to tell exactly the measure of the circle; that is to say, to know it with the same precision that we know, for example, that a parabolic segment is equal to two thirds of a parallelogram having the same base and the same altitude.

§ I.

The diameter of a circle being given; to find in approximate numbers, the circumference; or vice versa.

When moderate exactness only is required, we may employ the proportion of Archimedes, who has demonstrated that the diameter is to the circumference, nearly as 1 to $3\frac{1}{7}$, or as 7 to 22.

If we therefore make this proportion: as 7 is to 22, so is the given diameter to a fourth term; or if we triple the diameter and add to it a seventh, we shall have the circumference very nearly.

The circumference of a circle, the diameter of which is equal to 100 feet, will be found therefore to be 314 feet 3 inches $5\frac{1}{7}$ lines: the error in this case is about 1 inch 6 lines.

If we are desirous of approaching still nearer to the truth, we must employ the proportion of Metius, which is that of 113 to 355: we must therefore say, as 113 to 355, so is the given diameter to the required circumference. The same diameter as before being supposed, we shall find the circumference to be 314 feet, 1 inch, $10\frac{106}{113}$ lines: the difference between which and the real circumference is less than a line.

If still greater exactness be required, we have only to employ the proportion of 10000000000 to 31415926535; the error in this case, if the circumference were a great circle, such as the equator of the earth, would be, at most, half a line.

To find the diameter, the circumference being given, it is evident that the inverse proportion must be employed. We must, therefore, say as 22 is to 7, or as 355 to 113, or as 314159 is to 100000, or as 31415926535 to 10000000000, so is the given circumference to a fourth term, which will be the diameter required.

§ II.

The diameter of a circle being given; to find the area.

Archimedes has demonstrated, that a circle is equal to the rectangle of half the radius by the circumference. Find therefore the circumference, by the preceding paragraph, and multiply it by half the radius, or the fourth part of the diameter: the product will be the area of the circle, and the more exact, the nearer to truth the circumference has been found.

By employing the proportion of Archimedes, the error, in a circle of 100 feet diameter, will be about $3\frac{1}{8}$ square feet.

That of Metius would give an error less than 25 square inches, or about a sixth of a square foot. As the circle in question would contain about 7854 square feet, the error, at most, would be only one 47124th part of the whole area.

But the area of a circle may be found, without determining the circumference; for it follows, from the proportion of Archimedes, that the square of the diameter is to the area, as 14 to 11; from that of Metius, that it is as 452 to 355; from the proportion of 100000 to 314159, that it is as 100000 to 78539, or with still greater exactness as 1000000 to 785398.

The area of the circle therefore will be found by making this proportion, as 14 is to 11, or as 452 to 355, or as 1000000 is to 785398, so is the square of the given diameter, to a fourth proportional, which, if the last proportion has been employed, will be very near the truth.

§ III.

Geometrical constructions for making a square very nearly equal to a given circle, or a straight line equal to a given circular circumference.

Having shown some methods for finding numerically, and very near the truth, the proportion between a circle and the square of its diameter, we shall now give some geometrical constructions, exceedingly simple and ingenious, for accomplishing the same object.

1st. Let BADC (fig. 74 pl. 9) be a circle, of which AC is the diameter, and AB a quadrant; let AE, ED, and DC be chords equal to the radius; from the point B, draw to the points E and D the lines BE and BD, intersecting the diameter in F and G; the sum of the lines BF and FG, will be equal to the quadrant of the circle, within a five thousandth part.

2d. Let AD (fig. 75 pl. 9) be the diameter of the circle, C the centre, and CB the radius perpendicular to that diameter. In AD continued, make DE equal to the radius; then draw BE, and in AE continued make EF equal to it; if to this line EF, its fifth part FG be added, the whole line AG will be equal nearly, within a 17000th part, to the circumference described with the radius CA.

For if DA be supposed equal to 100000, AG will be found equal to 314153, with less than an unit of error: but the circumference corresponding to this diameter is, with the difference of nearly an unit, 314159; the error therefore at most is $\frac{6}{100000}$ of the diameter, or about the 17000th part.

3d. If the semicircle ABC (fig. 76 pl. 9) be given; from the extremities A and C of its diameter, raise two perpendiculars, one of them CE equal to the tangent of 30°, and the other AG equal to three times the radius; if the line GE be then drawn, it will be equal to the semi-circumference of the circle, within a hundred thousandth part nearly.

For it will be found by this construction, the radius being supposed to be 100000, that the line EG, within a unit nearly, is equal to 314162, and the semi-circumference would be, with the difference of nearly an unit, 314159; the error therefore is about $\frac{3}{100000}$ of the radius, or less than a hundred thousandth part of the circumference.

4th. Let A (fig. 77 pl. 9) be the centre of the given circle, and DE and CB its two diameters, perpendicular to each other. On any radius, such as AD, make AF equal to half the side EC of the inscribed square; draw BFI indefinitely, and to the point H, draw FH dividing AC in extreme and mean ratio, AH being the lesser segment; if CI be drawn parallel to FH through the point C, the square BLKI, constructed on BI, will be nearly equal to the circle of which BC is the diameter.

For it will be found by calculation, that BH and BF are respectively equal to 69098 and 61237, the radius being 100000; BI therefore will be found equal to 88623, the square of which is 78540 &c, the square of the diameter being 100000 &c, while the circle is 78539 &c.

5th. Inscribe in the given circle a square, and to three times the diameter add a fifth part of the side of the square; the result will be a line which will differ from the circumference by about a 17000th part only.

§ IV.

Several methods for making, either numerically or geometrically, and very near the truth, a straight line equal to the given arc of a circle.

1st. Let the given arc, which ought never to exceed 30°, be BG (fig. 78 pl. 9). To obtain the length of it very nearly in a straight line, draw BH perpendicular to the diameter AB, and continue the diameter to D, so that AD shall be equal to the radius; if DH be then drawn, it will cut off from BH the line BE somewhat less, but very nearly equal to the arc BG.

But if the line *df*G*e* be drawn in such a manner, that the segment *df*, intercepted between the circle and the diameter continued, shall be equal to the radius, the straight line B*e* will then be somewhat greater than the arc BG; but very near it, if the arc does not exceed 30 degrees.

For this theorem we are indebted to Snellius; but it was first demonstrated by Huygens. We shall show hereafter, that it is very useful in trigonometry.

2d. It has been demonstrated also by Huygens, that twice the chord of half an arc, plus the third of the difference between that sum and the chord of the whole arc, is nearly equal to the arc itself, when it does not exceed 30°.

For if we suppose the arc to be 30°, the chord will be 25882 parts, the diameter being 100000; that of half the same arc, or of 15°, will be 13053, the double of which is

26106; if from this we subtract 25882, the difference will be 224, the third of which, $74\frac{2}{3}$, added to 26106, will give $26180\frac{2}{3}$ for the arc of 30°. Twelve times this arc ought to give the whole circumference; but $26180\frac{2}{3}$, multiplied by 12, is equal to 314168, and the circumference is 314159, the difference, therefore, is only the nine hundred thousandth part of the radius.

REMARK.

As we promised to give a short account of the different attempts made respecting the quadrature of the circle, we shall here discharge our promise. What we are going to say on the subject is only an abstract from a very curious work, published by Jombert in 1754*.

It will first be proper to divide those who have employed themselves on this problem into two classes. The first, consisting of able geometricians, were not led away by illusions. Being aware of the difficulty or impossibility of the problem, they confined themselves merely to the finding out methods of approximation more and more exact; and their researches have often terminated in discoveries in almost every part of geometry.

The other class consists of those who, though scarcely acquainted with the elements of geometry, and scarcely knowing on what principles the problem depends, have made every effort to solve it, by accumulating paralogisms on paralogisms. Like the unfortunate Ixion, condemned to roll up a heavy burden eternally, without being able to bring it to the place of its destination, we find them twisting and turning the circle in every direction, without advancing one step further. When a geometrician has convinced them of an error in their pretended demonstrations, we see them returning a few days after, with the same demonstration in a new form, but equally contemptible.

* The author of that curious little work was Montucla himself.

Very often they do not hesitate to contest the best established truths in the elements of geometry, and in general sensible of the weakness of their knowledge in this department of science, they consider themselves as specially illuminated by Heaven to reveal truths to mankind, the discovery of which it has withheld from the learned, in order to confer the honour of it on idiots. Such is the ridiculous but real picture of this sort of men. It may be readily conceived that in the short history we are about to give of the quadrature of the circle, we shall not be so unjust towards the eminent geometricians, as to couple them with such visionaries. The singular flights of the latter will only furnish us, towards the end of this article, with matter for an amusing addition to it.

Geometry had scarcely been introduced among the Greeks, when the quadrature or measure of the circle began to give employment to all those who possessed a mathematical genius. Anaxagoras, it is said, exercised himself upon it while in prison; but with what success we are not informed.

The question had already become celebrated in the time of Aristophanes, and perhaps had made some geometrician lose his senses; for in order to ridicule the celebrated Meto, that comic writer introduces him on the stage, promising to square the circle.

Hippocrates of Chios certainly made it an object of his research; for it could be only by endeavouring to square the circle that he discovered his famous lunules. Some even ascribe to him a certain combination of lunules, from which, as they pretend, he deduced the quadrature of the circle; but in our opinion without any foundation; for as he held a distinguished place among the geometricians of his time, he could not be a dupe to the paralogism of a school-boy: his object was only to show, that if the lunule described on the side of an inscribed hexagon could be made equal to a rectilineal space, the quadrature of the circle could be thence deduced; and in this he was perfectly right.

It is very probable that geometricians were not long ignorant that the circle is equal to the rectangle of half the circumference by the radius. Before the time of Plato, geometry had been enriched with more difficult discoveries, yet this truth is first found in the writings of Archimedes. Something more however was necessary: the proportion between the circumference and the diameter, or the radius, remained to be determined; and this discovery occasioned, no doubt, many a sleepless night to that profound geometrician. Not being able to succeed with geometrical precision, he had recourse to approximation, and found, by calculating the length of an inscribed polygon of 96 sides, and that of a similar one circumscribed, that the diameter being 1, the circumference would be more than $3\frac{10}{71}$, and less than $3\frac{10}{70}$, or $3\frac{1}{7}$. For he showed that the inscribed polygon is somewhat greater than $3\frac{1}{70}$, and that the circumscribed is somewhat less than $3\frac{1}{70}$.

Since that time, if great exactness be not required, to find the ratio of the diameter to the circumference, the proportion of 1 to $3\frac{1}{7}$, or of 7 to 22, is employed; that is to say, the diameter is tripled and $\frac{1}{7}$ of it is added: this 7th is never neglected, but by the most ignorant workmen.

This object, we know, engaged the attention of several more of the ancient geometricians; among whom were Apollonius, and one Philo of Gadara; but the exactest approximations which they found have not reached us.

The first of the modern geometricians, who made any additions to what the ancients had transmitted to us, respecting the measure of the circle, was Peter Metius, a geometrician of the Netherlands, who lived about the end of the 16th century. Being employed in refuting the pretended quadrature of one Simon à Quercu, he found this very remarkable proportion, which approaches exceedingly near to the truth, between the diameter and the circumference, viz, as 113 to 355. The error is scarcely the ten millionth part of the circumference.

After him, or about the same time, Vieta, a celebrated French analyst and geometrician, expressed the ratio of the circumference to the radius by the proportion of 10000000000 to 31415926535, and showed that the latter number was too small, but that if its last figure were augmented by only one unit, it would be too great. About the same period also Adrian Romanus, a geometrician of the Netherlands, carried this approximation to 16 figures; but all these were far exceeded by Ludolph van Ceulen, a native of the Netherlands likewise, who carried this proportion to 35 figures, and showed that, if the diameter be unity followed by 35 ciphers, the circumference will be greater than 314159265358979323846264338327950288, and less than 314159265358979323846264338327950289. He was so proud of this labour, which however required less sagacity than patience, that, like Archimedes, he requested it might be inscribed on his tomb-stone: his desire was complied with, and this singular monument is still to be seen, it is said, in one of the towns of Flanders.

Willebrord Snell, another countryman of Metius, made several important additions to what had been done on this subject, in his book entitled *Cyclometria.* He discovered the method of expressing, by a very approximate proportion, and an exceedingly simple calculation, the magnitude of any arc whatever; and he made use of it to verify the calculation of van Ceulen, which he found to be correct. He then calculated a series of polygons, both inscribed and circumscribed, always doubling the number of sides, from the decagon to that of 5242880 sides; so that, when a proportion between the diameter and circumference of the circle, pretended to be exact, is proposed, one may refute it by means of this table, and show which is the circumscribed polygon greater than the supposed value of the circumference, and what circumscribed polygon it surpasses: in either case this will serve to prove the falsity of the pretended rectification of the circular circumference.

The celebrated Huygens, when very young, enriched the theory of the measure of the circle with a great many new theorems. He combated also the pretended quadrature of the circle, which father Gregory Saint Vincent, a jesuit of the Netherlands, announced as discovered, and requiring only a few calculations, which he dexterously forgot to make. Gregory Saint Vincent, however, was an able geometrician; he wrote an answer to Huygens, and the latter replied; some of Gregory's pupils entered the lists also, and another jesuit, a geometrician, combated on the same side. But it is certain, whatever father Castel may have said, that Gregory was mistaken, and that his large work, which contains some very ingenious things, ended with an error, or something unintelligible. As he pretended to have found the quadrature of the circle, why did he not perform those calculations which are necessary to express it numerically? But this was never done, either by him, or by any of his pupils, who carried on the dispute with a great deal of asperity.

James Gregory, a celebrated geometrician in Scotland, undertook, in 1668, to demonstrate the absolute impossibility of the quadrature of the circle. This he did by a very ingenious method of reasoning, which deserves perhaps to be better examined. However, it did not meet with the approbation of Huygens, and this produced a very warm dispute between these two geometricians. But it must be confessed that Gregory gave several very ingenious methods for approaching nearer to the measure of the circle, and even to that of the hyperbola.

The higher geometry supplies us with a great number of different methods for finding, by approximation, the measure of the circle, and the greater part of them are easier than the preceding; but this is not a proper place for entering into an explanation of them. We shall content ourselves with observing, that by means of these methods the approximation of Ludolph van Ceuler has been carried as far as 127 figures or decimals. Sharp, an English geometrician, first carried it to 74 figures; Mr. Machin extended it to a hundred, and M. de Lagny continued it to 127: it is as follows. If the diameter be unity, followed by 127 ciphers, the circumference will be greater than 31415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679821480865132723066470938446, and less than the same number, when the last figure is increased only by unity. The error therefore is less than a portion of the diameter expressed by unity, divided by unity followed by 127 ciphers. If we suppose a circle, the diameter of which is a thousand million of times greater than the distance of the sun from the earth, the error in the circumference would be a thousand millions of times less than the thickness of a hair.

It is even possible to go still further; and Euler has pointed out the method, in the Transactions of the Imperial Academy of Sciences at Petersburg; but it must be confessed that it would be superfluous labour.

We cannot conclude better this short history of the quadrature of the circle, than by an account, which will no doubt amuse some of our readers, of those who have miscarried in their attempts to solve this problem, or who have fallen into ridiculous errors on the subject.

The first, among the moderns, who pretended to have found the quadrature of the circle, was cardinal de Cusa. One of his methods was, to roll a circle or a cylinder over a plane, till the point which first touched it should touch it again; and he then endeavoured, by a train of reasoning, which displayed nothing geometrical, to determine the length of the line thus described. He was refuted by Regiomontanus, in 1464 or 1465.

After him, viz, about the middle of the 16th century, Orontius Finæus, though professor royal of the mathematics, rendered himself ridiculous by his paralogisms, not only in regard to the quadrature of the circle, but also in regard to the trisection of an angle and the duplication of the cube. Peter Nonius however, a Portuguese geometrician, and J. Borelli his former pupil, clearly exposed the fallacy of his reasoning. The same Orontius Finæus published also a work on Gnomonics, which is nothing but a series of paralogisms.

We are astonished to find the celebrated Joseph Scaliger fall, soon after, into the same error. As he had no great esteem for geometricians, he was desirous to show them the superiority of a man of letters, in solving, by way of amusement, what had so long difficulted them: he attempted the quadrature of the circle, and seriously imagined he had discovered it, by giving, as the measure of it, a quantity which was only a little less than the inscribed dodecagon. It was therefore no great difficulty for Vieta, Clavius, and others, to refute his reasoning: this threw him into a violent passion; and, according to the practice of that period, exposed the latter in particular to a great many epithets not very decent, while it confirmed Scaliger more and more in his opinion, that geometricians were destitute of common sense.

We are sorry to include, among this class, the celebrated Danish astronomer, Longomontanus, who pretended to prove that the diameter of a circle is to the circumference, exactly as 100000 is to 314185. Soon after, the famous Hobbes imagined also that he had found the quadrature of the circle, and being refuted by Dr. Wallis, he undertook to prove that the whole system of geometry before taught was nothing but a series of paralogisms.

This forms the subject of a work entitled, De ratiociniis et fastu Geometrarum.

Olivier de Serres, the agriculturist, by weighing a circle and a triangle, equal to the equilateral triangle inscribed, believed he had found that the circle was exactly the double of it. This weak man did not see that this double is exactly the hexagon inscribed in the same circle.

A. M. Dethlef Cluver pretended, in 1695, to have squared the circle; and he reduced the problem to one much easier, which he announced in the following manner: Invenire mundum Menti divinæ analogum. He unsquared the parabola, and endeavoured to prove that Archimedes had been deceived in regard to the measure of that figure.

Leibnitz endeavoured to engage him in a dispute with M. Nieuwentyt, who then started a great many difficulties against these new calculations; but the attempt did not succeed.

Though these ridiculous attempts, as appears, ought to have prevented others, men were seen, and are still seen daily, falling into errors of the like kind. About thirty years ago a M. Liger pretended that he had found out the quadrature of the circle, by demonstrating that the square root of 24 was the same as that of 25; and that of 50 the same as that of 49: this he demonstrated, according to his own terms, not by geometrical reasoning, which he abhorred, but by mechanism combined with figures.

A certain Sieur T de N—— found out something not less curious, viz, that curves ought not to be measured by comparing them with straight lines, but by comparing them with curves. This being once demonstrated, the quadrature of the circle is merely children's amusement.

M. Clerget made another discovery no less interesting, viz, that the circle is a polygon of a determinate number of sides: and he thence deduced, which is very curious, the magnitude of the point where two unequal spheres touch each other. He demonstrated also the impossibility of the motion of the earth. No one before him had been able to suspect the least affinity between these questions.

But what shall we say of the complex calculations of the late M. Basselin, a professor in a university, who, after as much labour almost as van Ceulen, found a proportion between the diameter and circumference beyond the limits even of Archimedes? This weak man, who had so happily discovered the quadrature of the circle, was ignorant, till some days before his death, that Archimedes had squared the parabola. He proposed also, had he recovered from his malady, to examine the process of Archimedes, being fully convinced that the geometrician of Syracuse had been deceived.

But if these men incurred only ridicule, and ridicule confined to the circle of a small number of geometricians, we are now going to introduce one to whom the ambition of squaring the circle cost much dearer. We allude to the Sieur Mathulon, who, from being a manufacturer of stuffs at Lyons, commenced geometrician and mechanist; but with less success than Hippocrates of Chios, who, from being a wine merchant at Athens, became an illustrious geometrician. Sieur Mathulon, about 40 years ago, deposited the sum of 1000 crowns at Lyons, and having announced to geometricians and mechanists the discovery of the quadrature of the circle and the perpetual motion, declared he would give the above sum to the person who should prove that he was in an error. M. Nicole, of the Academy of Sciences, proved that his knowledge of geometry was very limited; that his pretended quadrature was a mere paralogism; and demanded the 1000 crowns, which were adjudged to him. The Sieur Mathulon demurred, and maintained that he ought to prove also the falsity of his perpetual motion; but he lost his suit, and M. Nicole gave up the 1000 crowns to the general hospital at Lyons, to which they were delivered.

Had the Chatelet of Paris been equally severe, a similar folly would have cost much more to a man of some property, who, about 30 years ago, announced the quadrature of the circle; defied the whole world to refute him; and at last, by way of challenge, deposited 10000 livres to be adjudged to the person who should prove that he was mistaken. It is impossible, without lamenting the weakness of the human mind, to see this grand discovery reduced to dividing a circle into four equal parts, by perpendicular diameters, turning these quadrants with their four angles outwards, so as to form a square, and then pretending that this square is equal to the circle. According to the principles of this pretended mathematician, for two figures to be equal, it is not necessary that they should touch each other throughout their whole extent; it is sufficient that they touch or can touch. Thus the square is not only equal to the inscribed circle, but even to any figure included in the circle, the salient angles of which touch the circumference.

It would not have been difficult to show to any other person than the author, that this was absolute nonsense. Three persons appeared as claimants of the 10000 livres; the cause was tried at the Chatelet, but this tribunal was of opinion that a man's fortune ought not to suffer for the errors of his judgment, when these errors are not prejudicial to society. On the other hand, the king decreed that the bet should be considered as void; and that both parties should take back their money. The author extorted from the Academy of Sciences a sentence, by which he was desired to study the elements of geometry; but he was still convinced that future ages would blush for the injustice done to him by that in which he lived. Before we conclude this article, we must say a few words respecting M. le Rohberger de Vausenville, who in a work called Consultation sur la Quadrature du Circle, asks geometricians, whether the quadrature of the circle would not be found, if means could be devised for determining the centre of gravity of a sector of a circle, in common parts of the radius and the circumference of the same circle. We do not rightly understand what the author means by common parts of the radius and the circumference. If he means those parts of the radius in which it is usual to express the circumference, as when it is said that if the radius be 100, the circumference will be 314, we can answer, in the name of all geometricians, that the quadrature of the circle would, in that case, be found. We will even not hesitate to tell him, that in whatever manner he determines, in the axis of a sector or arc of a circle, its centre of gravity, provided that in this determination the arc itself is not employed as one of the data, he will have solved this famous problem; for who does not know that the distance of the centre of gravity of the semi-circumference, for example, from the centre, forms a third proportional to the fourth part of the circle and the radius? But this determination of the centre of gravity of the sector, or arc of a circle, is a discovery rather to be wished than hoped for.

M. de Vausenville had no need to challenge, either individually or in general, all the geometricians of Europe, and even those of Turkey and Africa, where the meaning of the words centre of gravity is certainly not known; and he had still less occasion to inform them that if they did not refute him, he would consider their silence as a sign of their defeat, and that his quadrature was acknowledged as resting on a solid foundation. This bravado will certainly excite neither the Eulers, the d'Alemberts, nor the Bernoullis, &c, to attack his quadrature. Either M. de Vausenville is right, and in that case mathematicians will acknowledge his discovery, and bestow on him every just praise; or his pretended quadrature is a mere paralogism, and of course it will meet with as little attention as that of Henry Sullamar, a real Bedlamite, who found it in the number 666, inscribed on the forehead of the beast in the

Revelations, or those of many others which deserve, in like manner, to be consigned to oblivion.

PROBLEM XLVIII.

Of the Length of the Elliptical Circumference.

We have spoken in a pretty full manner of the circular circumference, the exact determination of which in length would give the quadrature of the circle. But no author, as far as we know, has said any thing satisfactory, or useful in a practical view, respecting the circumference of the ellipse. It is however necessary, in many cases, and even in practical geometry, to know the length of that curve; in the higher geometry there are also a great many problems, the solution of which depends on the same knowledge: a few observations therefore on this subject may be of utility.

Some authors, who have written on practical geometry, are of opinion that the circumference of an ellipse is an arithmetical mean between the circumference of the circles described on the two axes as diameters; but this is a mistake; and had they possessed a little more of the spirit of geometry, they would have readily perceived it; for it may be easily demonstrated that this is false in an ellipse much elongated, as in that which has the greater axis 20, and the lesser 2. The circumference of this ellipse indeed will certainly be greater than 40, while the mean proportional between the circumferences of the circles described on its axis, as diameters, will be only 34½.

The rectification of the elliptical circumference is a problem which is almost the same, in regard to the quadrature of the circle, as the latter is to a common problem in geometry. John Bernoulli is the only person who has given a method susceptible of being reduced to practice for measuring the length of the elliptic line. He shows indeed in an excellent memoir, published in his works, how to determine the circular circumferences which are limits alternately less and greater than the circumference of a given ellipse: and by this method we have calculated the following table. We have supposed a series of ellipses, one half of the common greater axis of which is 10 parts, while the half of the less axis becomes successively 1, 2, 3, &c, as far as 10, the last value given by a circle; and we have found that the length of the circumferences of the ellipse is as here expressed.

Common length of the greater axis 20*

Lesser axis.	Length of the elliptic circumference.	Length of the mean circumference of the circles described on the greater and less axis.
2	40·63245	34·5579
4	42·01968	37·6990
6	43·68526	40·8406
8	46·02506	43·9822
10	48·44215	47·1238
12	51·05407	50·2654
14	53·82377	53·4070
16	56·72739	56·5486
18	59·81022	59·6902
20	62·83185	62·83185

It here appears, that the circumference of the circle, which forms a mean between those described on the greater and less axis, is always less than the elliptic line, and the more sensibly so, the more the ellipsis differs from a circle: in the first of the above ellipses the error is the 7th part.

By the help of this table all the mean lengths of the ellipsis between the preceding may be calculated: nothing is necessary but to take the proportional parts. Let us

* Sir Jonas Moore also calculated a like table for the elliptic circumferences, extending ten times as far, that is to a hundred different ellipses, the conjugate axes gradually increasing from 1 to 100, which is the constant transverse. The numbers indeed are set down to four decimals, but they are not commonly true to more than two. EDITOR.

suppose, for example, that the greater axis of a semi-ellipse is 20 feet, and that the half of its less axis is 7½ feet; it is evident that, in this case, the whole of the less axis will be 15 feet. This ellipsis then will hold a mean place between that in which half the less axis is $\frac{14}{20}$ of the greater, and another in which the less axis is $\frac{16}{20}$. But by dividing the difference between the lengths of these two ellipses into two equal parts, it will be found, without any considerable error, that the length of the circumference of the mean ellipse will be 55·27558 parts, the axis being 20; consequently the half of the proposed ellipsis having its transverse diameter equal to 20 feet, and its conjugate to 7½, will be 27 feet 6 inches and 8 lines: the error being scarcely a line.

PROBLEM XLIX.

To describe geometrically a circle, the circumference of which shall approach very near to that of a given ellipse.

It is to Mr John Bernoulli also that we are indebted for this simple and ingenious method of describing a circle isoperimetrous to a given ellipse. As it may serve as a supplement to what we have said on the rectification of the ellipse, we shall here give it a place.

Form the two axes of the given ellipsis into one straight line, as AD (fig. 130 pl. 16), in which AB is equal to the greater axis, and BD to the less: let this line AB be the diameter of a semi-circle AED, which must be divided into 4, 8, 16, or 32 parts, &c, at pleasure, and according as greater precision may be required. We shall here suppose the number of equal parts to be 16. From the point B, draw to each point of division straight lines; then take the 16th part of the sum of all these lines BA, B 1, B 2, B 3, &c, as far as BD inclusively, and if with the line hence arising as radius, a circle be described, you will have a circular circumference so nearly equal to that of the given ellipse, that it will not differ from it one hundred thousandth part, even in the most unfavourable case, such as that, for example, where the ratio of the axes of the ellipse is as 10 to 1.

It may be readily seen, that if the semicircle had been divided into 8 parts, it would have been necessary to take only the 8th part of the sum of all the lines drawn to the points of division, including the points D and A. And so for other numbers of parts.

If this operation were performed with a circle of a foot radius, the precision of the result would approach very near the truth; and by means of a geometric scale, with exceedingly minute divisions, a very satisfactory numerical approximation might be found without calculation*.

PROBLEM L.

To determine a straight line nearly equal to the arc of any curve whatever.

We shall suppose that the amplitude of the given arc is not very considerable, as not more than 20 degrees; that is to say, if tangents be drawn at the extremities of the arc, and then perpendiculars to these tangents, the angle included between these perpendiculars shall be at most 20 degrees.

* It is noticed above, by Montucla, that an arithmetical mean between the two axes of an ellipse, has been often taken as the diameter of a circle of equal circumference with the ellipse. It may be added that this rule always gives the perimeter in defect, or less than just.

Another rule, almost as easy, which gives the perimeter always in excess, or more than just, is this: Square each axis, and take the arithmetical mean between these squares, that is, add the squares together, and take half the sum; then extract the square root of this mean, and it will be nearly the diameter of a circle of equal circumference.

As this latter rule is nearly as much in excess as the former is in defect, if an arithmetical mean between them be taken, that is half their sum, it will be the diameter of a circle of equal circumference, within the fifty thousandth part of the whole, and is the nearest approximating rule yet given. These rules are taken from my treatise on Mensuration, where several others may also be seen, both for the whole circumference and also for any part of it. EDITOR.

This supposition being made; draw the chord of the arc; and then find, either by calculation, or by means of the compasses, the third of the tangents comprehended between the place where they meet and the points of contact; if we then add to this third two thirds of the chord, we shall have a straight line so nearly equal to the arc, that in the present case the difference will be but a ten thousandth part. But if the amplitude be only about 5 degrees, the error will not be a millionth part, as has been shown by M. Lambert, member of the Academy of Sciences at Berlin, in a very interesting work, published in German, which is highly worthy of being translated.

If the amplitude of the given arc be greater, as about 50 degrees for example, nothing will be necessary but to divide it into three parts nearly equal, and to draw tangents to the extremities of the arc and to the points of section, which will give a portion of the polygon circumscribed about the curve; if the three chords of the three parts of the arc be then drawn, and if two thirds of these three chords be added to the third of the tangents, forming the circumscribed polygon, the result will be a line equal, within a hundredth thousandth part, to the length of the given arc.

PROBLEM LI.

A circle, having a square inscribed in it, being given; to find the diameter of a circle in which an octagon of a perimeter equal to the square can be inscribed.

Let AB (fig. 79 pl. 10) be the diameter of the given circle, and AD the side of the inscribed square. Divide AD into two equal parts in E, and raise EF perpendicular to AD, meeting the given circle in F; if AF be then drawn, it will be the diameter of a circle, in which if an octagon be inscribed, it will be equal in perimeter to the given square.

For it is evident that the circle described on the diameter AF, will pass through the point E, since the angle AEF is a right angle. It is also evident that the line drawn from I, the centre of the second circle, to the point E, will be parallel to DF, because the sides AD and AF of the triangle DAF are bisected in the points E and I. But the angle AFD is half a right angle, being half of DCA, which is a right angle, since the chord of the inscribed square subtends an arc of 90°: consequently the angle AIE is equal to 45°; whence it follows that AE is the side of the octagon inscribed in the circle having AF for its diameter. And it is evident that 8 times AE is equal to 4 times AD.

REMARK.

If AE be, in like manner, divided into two equal parts in G, and if GH be drawn perpendicular from the point G, till it meet the second circle; by drawing AH, that line will be the diameter of a third circle, in which if a polygon of 16 sides be inscribed, it will be isoperimetrous to the above square or octagon.

Hence it follows, that if this operation were infinitely continued, we should obtain a circle or polygon of an infinite number of sides, isoperimetrous to a given square. The circumference of this circle therefore would be equal to the perimeter of the square, and we should thus have the quadrature of the circle.

We have seen a very ingenious attempt to discover the quadrature of the circle on this principle. The author M. Janot, professor of mathematics in the Royal Military School, reduced the problem to a very exact equation, but complex, by the solution of which he expected to obtain this last diameter; but when he seriously tried to reduce it, he found the two members of his equation to be composed of the same terms, which of course gave him no solution.

PROBLEM LII.

The three sides of a rightangled triangle being given; to find the value of its angles without trigonometrical tables.

We shall first suppose that the ratio of the hypothenuse to the least side is greater or not less than 2 to 1, in order that the angle opposite to that side may be at most about 30°; for the error will be less the more that angle is below 30°.

This being premised; let us suppose, for example, that the hypothenuse of the triangle is equal to 13, the greater side comprehending the right angle 12, and the less 5. We must then make this proportion: as twice the hypothenuse, plus the greater side, or 38, is to the less side or 5, so is three times unity or 3, to a fourth proportional, which will be $\frac{15}{38}$. But $\frac{15}{38}$ reduced to a decimal fraction is 0·39473: if this number be divided by 0·1745, the quotient will be the number of the degrees and parts of a degree contained in the angle opposite to the less side. This quotient is $22\frac{621}{1000}$, which makes 22° 37′ 15″. By the tables it will be found to be 22° 37′ 28″.

If the sides of the triangle are nearly equal, such for example as 3, 4, 5, we must suppose in the triangle a line CD (fig. 80 pl. 10), dividing the angle opposite to the side AB, or that represented by 3, into two equal parts. But it is known that in this case the opposite side AB will be divided in the same ratio as the adjacent sides; consequently the segment BD may be found by the following analogy:

As the sum of the two other sides or 9, is to 3, the third side, so is CB or 4, to BD, which will be $\frac{12}{9}$ or $\frac{4}{3}$; if the squares $\frac{4}{3}$ and 4, or of CB and BD, be then added together, by extracting the square root of the sum, which in decimals is 17·777, we shall have for the square root 4·21637, which will be the value of CD. In the last place, by applying the above rule to the triangle BCD, we shall find the angle

BCD to be 18° 26′ 7″, and consequently its double, or the angle ACB, 36° 52′ 14″. By trigonometrical tables the latter will be found to be 36° 52′ 15″; so that the difference is only one second.

PROBLEM LIII.

An arc of a circle being given, in degrees, minutes, and seconds; to find the corresponding sine, without the help of trigonometrical tables.

The solution we are going to give of this problem, is not so simple and short as the preceding; but it appears to be the best hitherto proposed, especially as it is easy, and may be readily remembered by means of an observation we shall make at the end, and which will show its source as well as the demonstration of it.

In this problem there are three cases, which require three different methods of operation. The given arc may exceed 60°, or it may be less or at most not more than 30°; and in the last place it may be greater than 30°, but less than 60°.

1st. We shall suppose that the arc exceeds 60°, and that its sine is required. Take its complement to 90°, and reduce that arc into parts of the radius, which we shall suppose to be 100000; for this purpose, nothing is necessary but to multiply the degrees it contains by $1745\frac{4}{10}$, and the minutes by 29·09, and then to add the products. Square this arc thus reduced, and raise it also to the 4th power; divide the square of it by 2, and from the quotient subtract unity or the radius: divide the 4th power of it by 24, and add the quotient to the above remainder: the number thence resulting will be nearly the sine of the given arc.

Let the given arc, for example, be 70° 30′; its complement to 90° is 19° 30′, which reduced to parts of the radius, as beforesaid, will give 34025. The square of this number, suppressing the five last figures, which are useless, because we have no occasion for more than 100000 parts of the radius, is 11583, and its half 5792, which taken from 100000 leaves 94208. Square 11583, which will give the 4th power of 34035; and if five figures be suppressed, as useless for the reason before mentioned, we shall have 1341, which must be divided by 24. The quotient, which is somewhat less than 56, being added to 94208, will make 94264, which will be the sine of 70° 30′. And this is exactly what it will be found to be in the tables of sines.

2d. Let us now suppose that the given arc is at most 30°. Find the cube and 5th power of that arc reduced to parts of the radius; then divide the cube by 6, and the 5th power by 120; if the first quotient be subtracted from the arc, and the second be added to the remainder, we shall have the value of the sine, a very small error excepted.

Let the given arc, for example, be 30°. When reduced to 100000th parts of the radius it will give 52362, the cube of which, suppressing the last ten figures, will be 14354. The 6th part of this number is 2392, which taken from the arc 52362, leaves 49970. The 5th power of the same number 52362, suppressing the last twenty figures, is 3935, which divided by 120 gives 32; if 32 be added to the above remainder, we shall have 50002 for the sine of 30°; which it is well known is exactly 50000; consequently the error is only two units in the last figure.

3d. If the arc be between 30° and 60°, for example 45°; take the difference between that arc and 60° which is 15°, and add it to 60°; the sum will be 75°, the sine of which must be found by the first rule. Then find that of 15° by the second; and subtract it from that of 75°; the remainder will be the sine of 45°; for, according to a theorem in trigonometry, the sines of two arcs, equally distant from 60°, have for difference the sine of that arc by which each of these two arcs differs from that of 60°.

If, instead of the sine of an arc, that of its complement be required, the same rules may be employed: the sine of the complement of 20°, for example, is the right sine of 70°; and, on the other hand, the sine complement of 70°, is the right sine of 20°; by which it may be readily seen, that to find the sine complement of an arc, nothing is necessary but to find the right sine of the complement of the arc.

When the right sine and the sine complement of an arc are known, it will be easy to find the tangent by the following proportion: As the sine complement, or cosine, is to the sine, so is radius to the tangent: nothing therefore is necessary, but to divide the sine, increased with any number of ciphers at pleasure, by the cosine.

REMARK.

We have here given a method for supplying the place of tables, so necessary in practical trigonometry, or of forming them very expeditiously, in cases when they are not at hand, or cannot be procured. I was once myself in such a situation, having lost my baggage, which was taken from me by a party of the Iroquois Indians, when posted at Oswego in Canada. In that dreary abode, I endeavoured to amuse myself by the study of geometry. An opportunity of performing some trigonometrical operations occurred; but, being destitute of books, I fortunately remembered the theorem of Snellius, which serves as a basis for the solution of the preceding problem: in short, I recollected two expressions, in infinite series, which give the value of the sine and cosine, the arc being given. The first, as is well known, a being made to represent the arc, is $a - \frac{a^3}{6} + \frac{a^5}{120} - \frac{a^7}{5040}$ &c, and the second $1 - \frac{a^2}{2} + \frac{a^4}{24} - \frac{a^6}{720}$ &c. But when the arc a is very much below the value of the radius or unity, it is evident that the three first terms of each will be sufficient, because all the following, terms become excessively small. After what has been said, the demonstration of these rules may be easily discovered.

PROBLEM LIV.

A circle and two points being given; to describe another circle, which shall pass through these points, and touch the former circle.

It is here evident, that these two points must be both within, or both without the given circle.

Let the two given points then be A and B, as in the two figures 81 and 82, pl. 10. Join these points by a straight line AB; and through one of them, for example A, and the centre of the given circle, draw the straight line AIH, intersecting it in the two points H and I; then take AD a 4th proportional to AB, AH, AI, and from the point D, draw the two tangents DE, D*e*; lastly, from the point A, through the two points of contact draw the two lines EAF, *e*A*f*, intersecting the circle in F and *f*: the circle described through the two points A and B, and through F, will touch the given circle in F; and if one be described through the points A, B, and *f*, it will touch the given circle in *f*.

PROBLEM LV.

Two circles and a point being given; to describe a third circle, which shall pass through the given point, and touch the other two circles.

Let the centres of the two given circles be the points A and C, (fig. 83 pl. 10) and their radii AB, CD. In the line which joins their centres continued, find the point F, the tangent from which to one of the circles shall be the tangent to the other, (by Prob. XII), and join the point F to the given point E: then make FG a 4th proportional to FE, FB, FD, and by the preceding problem, through the points G and E describe a circle which shall touch one of the two circles AB or CD: this third circle will touch also the other circle.

PROBLEM LVI.

Three circles being given; to describe a fourth, which shall touch them all.

It may be readily seen that this problem is susceptible of a great number of different cases and solutions; for the required circle may contain the three given circles, or only two of them, or even one, or the given circles may all be without it. But, for the sake of brevity, we shall confine ourselves to one of these cases, that where the circle to be described must leave the other three without it.

Let the three given circles then be denoted by A, B, C, (fig. 84 pl. 10) and let their radii be A*a*, B*b*, C*c*; let A also be the greatest, B the mean one, and C the least. In the radius A*a* make *ad* equal to C*c*, the radius of the least circle, and from A as a centre, with the radius A*d* describe a new circle. In the radius B*b*, make *be* equal to C*c* also, and from B as a centre, with the radius B*e* describe another circle; then, by the preceding proposition, through the centre C describe a circle, which shall touch the two new circles; let its centre be E, and its radius EG; diminish this radius by the radius C*c*, and from the same centre E describe another circle, which will evidently touch the three first circles given.

For since the circle described from the centre A with the radius A*d*, is within the proposed circle A, by the quantity *ad* or C*c*, it is evident that if the radius EG be diminished by that quantity, the circle described with this new radius, instead of touching the interior circle, having A*d* for its radius, will touch the proposed circle, the radius of which is A*a*.

It may be seen also that the same circle described with the radius EG, less C*e*, will touch externally the circle which has for its radius B*b*. Lastly, it will touch externally the circle having C*c* for its radius; consequently it will touch them all three externally.

REMARK.

This problem had some celebrity among the ancients; and indeed it is attended with a certain degree of difficulty. It terminated a treatise of Apollonius, entitled De Contactibus, which has been lost, but which Vieta, a celebrated geometrician who lived about the end of the 16th century, restored, and which may be found in his works, printed in Latin, at Leyden, in 1646, in folio, with the title of Apollonius Gallus, seu exsuscitata Apollonii Pergæi de Tactionibus Geometria.

Newton has given a beautiful and ingenious solution of this problem; but that of Vieta appeared to us preferable for the present work, being founded on easier principles. We cannot omit this opportunity of observing, that the above work of Vieta is a most elegant piece of geometry, treated in the manner of the ancients.

PROBLEM LVII.

What bodies are those, the surfaces of which have the same ratio to each other as their solidities?

This problem was proposed, in the form of an enigma, in one of the French Journals, entitled the Mercury, of the year 1773.

Reponds-moi, d'Alembert, qui découvre les traces
Des plus sublimes vérités ;
Quels sont les corps dont les surfaces
Sont en même rapport que leurs solidités ?

We do not find that d'Alembert condescended to answer this problem, for it may be readily seen, by those in the least acquainted with geometry, that two bodies well known, the sphere and the circumscribed cylinder, will solve it. Archimedes demonstrated long ago, that the sphere is equal to two thirds of that cylinder, both in surface and solidity, provided the two bases of the cylinder are comprehended in the former; and this is the answer which was given to the enigma in the following Mercury.

But we may go a little further, and say, that there are a great number of bodies which, when compared with each other, and with the sphere, will answer the problem also: such are all solids formed by the circumvolution of a plane figure circumscribed about the same sphere, and even all plane-faced solids, regular or irregular, that can be circumscribed about the same sphere; for the solidity of all these bodies is the product of their surfaces by the third of the radius of the inscribed sphere, while the solidity of the sphere is the product of its surface by the third of its radius.

Thus, the equilateral cone is to the inscribed sphere, both in surface and solidity, as 9 to 4.

The case is the same in regard to the sphere and the circumscribed isosceles cone; except that the ratio, instead of 4 to 9, will be different according to the elongation or oblate form of the cone.

If the sphere and the circumscribed cylinder possess this property, it is because the latter is a body produced by the circumvolution of the square circumscribed about the great circle of the sphere, on an axis perpendicular to two of the parallel sides.

If the square and inscribed circle revolved around the diagonal of the square, the surface and solidity of the body, thus produced, would be to each other as $\sqrt{2}$ is to 1.

We shall here propose a similar problem:

What are those figures, the surfaces and perimeters of which are to each other in the same ratio?

The answer is easy: the circle and all polygons, regular or irregular, circumscriptible of it.

THEOREM VIII.

The dodecagon inscribed in the circle is $\frac{3}{4}$ of the square of the diameter, or equal to the square of the side of the inscribed triangle.

This theorem, which is exceedingly curious, was first remarked by Snellius, a Dutch geometrician.

Let AC (fig. 85 pl. 10) be the radius of a circle, in which is inscribed the side AB of the hexagon; and if AD and DB be the sides of the regular dodecagon, it thence follows that, by drawing the radius DC, it will cut the side AB perpendicularly, and divide it into two equal parts. But it is evident, that the area of the dodecagon is equal to 12 times one of the triangles ADC, or DCB; and as the triangle ADC is equal to the product of the radius by the half of AF, or by the 4th part of the radius, that is to say, is equal to a 4th of the square of the radius, the twelve will be equal to 3 times the square of the radius, or to $\frac{3}{4}$ of the square of the diameter.

On the other hand, the side of an equilateral triangle inscribed in a circle, the diameter being unity, is equal to $\sqrt{\frac{3}{4}}$; consequently its square is also equal to $\frac{3}{4}$ of the square of the diameter, or to the dodecagon.

REMARK.

Two of the inscribed polygons, viz. the square and dodecagon, possess the property of having a numerical ratio to the square of the diameter; for the inscribed square is exactly the half; but of the regular polygons circumscribed, this property belongs only to the square.

Irregular polygons however, and even a great variety of them, commensurable to the square of the radius, might be inscribed in a given circle.

Let the diameter of the circle, for example, be 1; and let the four sides of the inscribed quadrilateral be $\frac{6}{10}$, $\frac{8}{10}$

$\frac{12}{71}$, $\frac{5}{13}$: its surface will be rational, and equal to $\frac{3028}{8410}$ of the square of the diameter.

PROBLEM LVIII.

If the diameter AB (fig. 86 pl. 10) *of a semicircle* ACB, *be divided into any two parts whatever,* AD *and* DB; *and if on these parts as diameters there be described two semicircles* AED *and* DFB, *a circle is required equal to the remainder of the first semicircle.*

From the point D raise DC perpendicular to AB, till it meet the semicircle ACB: if a circle be then described having DC for its diameter, it will be that required.

The demonstration of this problem, so well known, is deduced from a theorem in the 2d book of the Elements of Euclid, viz. that the square of AB, is equal to the squares of AD and DB and twice the rectangle of AD and DB; a rectangle to which the square of DC is equal by the property of the circle. Instead of these squares, if we substitute semicircles, which are in the same ratio, the problem will be demonstrated.

PROBLEM LIX.

A square being given; to cut off its angles in such a manner, that it shall be transformed into a regular octagon.

Let the given square be ABCD (fig. 87 pl. 10). In the two sides DC and DA, which meet in D, take any two equal segments whatever, DI and DK, and draw the diagonal IK; make DL equal to twice DK, plus the diagonal IK, and draw LI; if CM be then drawn parallel to LI, through the point, C, it will cut off from the side of the square the quantity DM, to which if DN be made equal, by drawing the line NM, we shall have the side of the octagon required. If AE, AF, BG, BH, CN and CO be made equal to the line DM, by drawing EF, GH and ON, the required octagon will be completed.

REMARK.

The solution above given is an example of what often happens in employing the algebraic calculus in the solution of geometrical problems; for there is a solution much more simple, and that of a nature to be self-evident even to a beginner. It is as follows:

Draw the diagonal AC (fig. 131 pl. 17) of the square, and also EF bisecting the opposite sides in E and F, and the diagonal in G. Draw GH so as to bisect the angle CGE; so shall EH be half the side of the octagon. Therefore make CI, BK, BL, AM, AN, DO, DP each equal to CH, and the angles of the octagon will be found.

PROBLEM LX.

A triangle ABC *being given; to inscribe in it a rectangle, in such a manner, that* FH *or* GI *shall be equal to a given square.*

On the base BC (fig. 88 pl. 11) describe the rectangle BD, equal to the given square, and let E be the point where AC is intersected by the side of this rectangle parallel to BC. On AC describe a semicircle, and having raised the perpendicular EL, till it meet the circumference, draw CL; on CK, the half of AC, describe also a semicircle, in which make CM equal to CL: if KF and KG be then made equal to KM, we shall have the points F and G, through which if two lines be drawn parallel to the base till they meet AB, and also two other lines perpendicular to the base, they will form the rectangles FH and GI, equal to each other, as well as to the rectangle DB, which was equal to the given square: therefore &c.

PROBLEM LXI.

Through a given point D (fig. 89 pl. 11) *within an angle* BAC, *to draw a line* HI, *in such a manner, that the triangle* IHA *shall be equal to a given square.*

Through the point D draw LE parallel to one of the sides of the given angle, and make the rhombus LEGA, equal to the given square. On the line DE describe a semicircle, in which apply DF equal to DL, and draw EF; lastly, if GH be made equal to EF, and HDI be drawn through the point H, the line HDI will be the one required.

PROBLEM LXII.

Of the Lunule of Hippocrates of Chios.

Though the quadrature of the circle be in all probability impossible, means have been devised to find certain portions of the circle which are demonstrated to be equal to rectilineal spaces. The oldest instance of a circular portion, which may be thus squared, is that of the lunules of Hippocrates of Chios; the construction of which is as follows:

Let ABC (fig. 90 pl. 11) be a right-angled triangle, on the hypothenuse of which describe the semicircle ABC, touching the right angle B: if semicircles be then described on the sides AB and BC, the spaces in the form of a crescent, AEBHA and BDCGB, will together be equal to the triangle ABC.

For it is well known that the semicircle on the base AC is equal to the two semicircles AEB and BDC, because circles are to each other as the squares of their diameters: if the segments AHB and BGC, which are common to both, be taken away, there will remain, on the one hand, the triangle ABC, and on the other the two spaces in the form of a crescent, AEBH and BDCG, and these remainders will be equal: therefore &c.

If the sides *ab*, *bc* are equal, as in fig. 91, the two lunules will evidently be equal, and each will be half of the triangle *abc*, that is to say, equal to the triangle *bfa* or *bfc*.

Hence we obtain a simpler construction of the lunule of Hippocrates. Let ABC (fig. 92) be a semicircle on the diameter AC, and AFC an isosceles right-angled triangle.

If from the point F as a centre, there be described through A and C, the arc of a circle ABC on the base AC, the lunule ABCD will be equal to the triangle CAF.

Since the square of FC is double the square of EC, or of EF, the circle described with the radius FC will be double that described with the radius EC: consequently a 4th part of the former, or the quadrant FADC, will be equal to the half of the second, or to the semicircle ABC. If the common segment ADCEA therefore be taken away, the remainders, viz, the triangle AFC, on the one hand, and the lunule ABCDA, on the other, will be equal.

REMARKS.

We shall take this opportunity of making the reader acquainted with several curious observations, added by modern geometricians to the discovery of Hippocrates.

1st. From the centre F (fig. 93) if there be drawn any straight line whatever FE, cutting off a portion of the lunule AEGA; that portion will still be quadrable, and equal to the rectilineal triangle AHE right-angled at H. For it may be easily demonstrated, that the segment AE will be equal to the semi-segment AGH.

2d. From the point E, if EI be let fall perpendicular on AC, and if FI and FE be drawn, the same portion of the lunule AEGA will be equal to the triangle AFI. For it may be easily demonstrated, that the triangle AFI is equal to the triangle AHE.

3d. The lunule therefore may be divided in a given ratio, by a line drawn from the centre F: nothing more is necessary than to divide the diameter AC in such a manner, that AI shall be to CI in that ratio; to raise EI perpendicular to AC, and to draw the line FE: the two segments of the lunule AGE and GEC will be in the ratio of AI to IC.

All these remarks were first made by M. Artus de Lionne, bishop of Gap, who published them in a work entitled Curvilineorum Amænior Contemplatio, 1654, 4°, and afterwards by other geometricians.

4th. If the two circles, forming the lunule of Hippocrates, be completed, the result will be another lunule, which may be called Conjugate, and in which mixtilineal spaces, absolutely quadrable, may be found.

From the point F, if there be drawn any radius FM, intersecting the two circles in R and M; we shall have the mixtilineal space RAMR, equal to the rectilineal triangle LAR: which can be easily demonstrated; for it may be readily seen that the segment AR, of the small circle, is equal to the semi-segment LAM of the greater.

Hence it follows, that if the diameter *mo* touch the small circle in F, the mixt triangular space ARF*m*A, will be equal to the triangle ASF, right-angled in S, or to half the lunule AGCBA.

5th. There are also some other portions of the lunule of Hippocrates that are absolutely squarable; which, as far as we know, were never before remarked. Let ABCFA (fig. 94) be a lunule, and let AB be a tangent to the interior arc. Draw the lines EA and *e*A making with AB equal angles; if from the point B there be then drawn the chords BE, B*e*, which will be equal, we shall have the mixtilineal space, terminated by the two circular arcs, EB*e*, AGF, and the straight lines A*e* and FE, equal to the rectilineal figure *e*AEB*e*.

This would be true, even if the figure ABCFA were not absolutely quadrable; that is to say, though ABC should not be a semicircle, provided the two circles were always in the ratio of 2 to 1.

PROBLEM LXIII.

To construct other lunules, besides that of Hippocrates, which are absolutely squarable.

The lunule of Hippocrates is absolutely squarable, because the chords AB, BC (fig. 92) and AC are such, that the square of the last is equal to the squares of the other two; so that by describing on the last an arc of a circle, similar to those subtended by AB and BC, the two segments AB and BC are equal to ADC.

This method of considering the lunule of Hippocrates conducts us to more general views; for we may conceive in a circle any equal number of chords at pleasure, for example four, as AB, BC, CD and DE, (fig. 95) of such a nature, that by drawing the chord AE, the square of it shall be quadruple of one of them; or more generally, the number of these chords being *n*, the square of AE may be to that of AB, as *n* to 1. Thus, if we describe on AE an arc similar to those subtended by the chords AB, BC, &c, the segment AE will be equal to the segments AB, BC, &c, together: if from the rectilineal figure ABCDE therefore, we take away the segment AE, and add to it the segments AB, BC, &c, the result will be a lunule formed of the arcs ACE and AE, which will be equal to the rectilineal polygon ABCDE.

The question then is, to resolve the following geometrical problem:

In a given circle to inscribe a series of equal chords, AB, BC, CD, &c, *in such a manner, that the square of the chord* AE, *by which they are all subtended, shall be to the square of one of them, as the whole number of them is to unity; triple, if there are three; quadruple, if there are four, &c.*

But we shall confine ourselves to cases that can be constructed by the elements of geometry, which will still give us two lunules, similar to those of Hippocrates, the one formed by circles in the ratio of 1 to 3, and the other by two circles in that of 1 to 5, besides two other lunules, formed by circles in the ratio of 2 to 3 and of 3 to 5.

Construction of the first lunule.

Let AB (fig. 96) be the diameter of the lesser circle, with which the lunule is to be constructed. Continue AB to D, so that BD shall be equal to the radius, and on AD as a diameter describe the semicircle AED, cutting BE, drawn perpendicular to AD, in E; draw DE, and make DF equal to it: on AF describe also a semicircle AHF, intersecting the radius CG, perpendicular to AB, in H; draw AH, and in the given circle make the chords AI, IK and KL equal to AH; then draw AL, and on that chord, with a radius equal to DE, describe an arc of a circle AL; by these means we shall have the lunule AGBLA, equal to the rectilineal figure AIKLA.

Construction of the second lunule, where the circles are as 1 to 5.

Continue the diameter of the given or less circle, till the part PD (fig. 97) be equal to half the radius; and draw the indefinite line DE perpendicular to AD; then from the point S, which divides the radius AC into two equal parts, with a radius equal to 3 times AC, describe an arc of a circle, cutting the before-mentioned perpendicular in E: make EF equal $\frac{1}{4}$ of AC, and DH equal to the radius; divide HF into two equal parts in G; and from G as a centre, with a radius equal to GH, describe an arc of a circle cutting the straight line AD in I: then make DK equal to HI, and draw KR perpendicular to the diameter, intersecting the semicircle described on AC in L; lastly, draw AL, and let the chords AM, MN, NO, OP, PQ, be made equal to it: if an arc of a circle be then described on AQ, with a radius equal to DE, the lunule ANPQA will be equal to the rectilineal figure AMNOPQA.

Lunules absolutely squarable may therefore be formed with circles, which are to each other in the ratios of 1 to 2, 1 to 3, and of 1 to 5. But there are no others formed by circles in simple multiple or sub-multiple ratio, which can be constructed merely by the rule and compasses. Those which might be formed with circles in the ratio of

1 to 4, 1 to 6, or 1 to 7, &c, would require the assistance of the higher geometry. The trisection of an angle, or the finding of two mean proportionals, is a problem of the same nature, and of the same degree, and to be solved only by the same means. But there are still two, which may be constructed by the help of simple geometry, and which are formed by circles in the ratio of 2 to 3 and of 3 to 5. For the sake of brevity, we shall confine ourselves to showing the method of construction.

For the first. Let there be any circle, the radius of which is supposed to be 1; inscribe in it a chord AB (fig. 98 pl. 12), equal to $\sqrt{\frac{9}{4} - \sqrt{\frac{33}{16}}}$, and let it be twice repeated, from B to C, and from C to D: draw the chord AD, and on AD describe an arc similar to the arc ABC; if the two equal chords AE and ED be then drawn, the lunule ABCDEA will be equal to the rectilineal polygon ABCDEA.

For the second. In a circle, the radius of which is 1, inscribe a chord equal to $\sqrt{\frac{5}{2} - \frac{1}{2}\sqrt{\frac{5}{3}} - \sqrt{\frac{5}{3} - \frac{1}{2}\sqrt{\frac{5}{3}}}}$, and carry it round five times; draw the chord of the quintuple arc, and describe on it an arc with a radius $= \sqrt{\frac{5}{3}}$; in this arc inscribe the three chords of its three equal parts, which may be done by common geometry, because each of these thirds is similar to a fifth of the first arc already given. We shall then have a lunule equal to a rectilineal figure, formed by the five chords of the small circle and the three chords of the greater.

PROBLEM LXIV.

A lunule being given; to find in it portions absolutely quadruble, provided the circles by which it is formed are to each other in a certain numerical ratio.

Let ABCDA (fig. 99, 100, 101) be a lunule, formed by two circles in any of the above ratios, ABC being a portion of the lesser circle, and ADC of the greater. Draw AE the tangent of the arc ADC, and then draw the line AF in such a manner, that the angle EAC shall be to the angle FAC in the same ratio as the less circle is to the greater; then one of the three following things will take place: AF will be a tangent to the circle ABC, fig. 99; or it will cut it, as in *f*, fig. 100; or as in *φ* fig. 101.

In the first case, the lunule will be absolutely squarable, and equal to the rectilineal figure KALC (fig. 99).

In the 2d, this lunule, minus the circular segment A*f*, will be equal to the rectilineal figure A*f*KCLA, or the space AKCL, plus the triangle AK*f* (fig. 100).

In the 3d, the same lunule, plus the circular segment *a φ* will be equal to the rectilineal space *aφ*K*cla* or the space *a*K*cl* minus the triangle *a*K*φ* (fig. 101).

We omit the demonstration, both for the sake of brevity, and because it may be easily conceived from what has been already said.

Hence it may be readily seen, that if the given circles have to each other a certain ratio, which will admit of the angle FAC (fig. 99, 100) being constructed with the rule and compasses, in such a manner as to be to the angle EAC, in the reciprocal ratio of these circles, we may draw the line FA, which will cut off from the lunule the portion ADCB*f*A, equal to an assignable rectilineal space. Now this will always be the case when the less circle is to the greater in the ratio of 1 to 2, or 1 to 3, or 1 to 4, or 1 to 5, &c; for the angle FAC must then be double, triple, quadruple, or quintuple of EAC: in this there is no difficulty. The case will be the same if the less circle is to the greater in the ratio of 2 to 3, or 2 to 5, or 2 to 7, &c; or the arc ADC, being susceptible of geometrical trisection, as is often the case, if the greater circle be to the less as 3 to 4, or 3 to 5, or 3 to 7, &c.

Another Method. Let AF (fig. 102) be a tangent to the circle ABC in A, and AE a tangent to the arc ADC in the same point. Draw the line AG in such manner, that the angle FAG may be to the angle EAG, as the greater circle is to the less; that is, that the angle FAE shall be to EAG, as the greater circle minus the less is to the latter: the line AG will then fall either on AC, or above it, as in AG, or below it, as in A*g*.

Now, in the first case, it may be easily demonstrated, that the lunule is absolutely squarable.

In the 2d, it may be shown that the same lunule, minus the mixtilineal triangle MGCM, is equal to an assignable rectilineal space.

In the 3d, it may be proved that the same lunule, if the mixtilineal triangle C*mg* be added, will be equal to the same rectilineal space.

Lastly, let there be drawn, in each of the preceding figures, between AC and AE (fig. 99, 100, 101, 102), any line whatever AN, forming with the tangent AE any angle NAE; then, in the angle FAE, draw another line A*n*, in such a manner, that the angle *n*AE shall be to EAN, as FAE to CAE. It may still be demonstrated, that the mixtilineal figure formed of the two arcs N*n*, AP, and the two lines A*n*, PN will be equal to a rectilineal space; which may be found, by dividing the arc N*n* into as many parts, similar to the arc AP, as the number of times that the less circle is contained in the greater: this may be performed geometrically, if the ratio of the one circle to the other be as 1 to 2, or 1 to 3, or 1 to 4, &c. If we here suppose it, for example, to be as 1 to 3, we shall have three equal chords *no*, *o*E, EN, and the portion of the lunule in question will be equal to the rectilineal figure A*no*ENA, since the three segments *no*, *o*E, &c, are together equal to the segment AP.

REMARK 1. We have also proposed and solved the following problem:

A lunule, not squarable, but formed by two circles in the ratio of 1 *to* 2, *being given; to intersect it by a line parallel* *to its base, which shall cut off from it a portion absolutely squarable.*

REMARK 2. The following method for dividing circles, &c, is so curious, that it is well deserving of a place here, in addition to the foregoing ways of dividing them into certain portions.

To divide Geometrically Circles and Ellipses into any Number of Parts at pleasure, and in any proposed Ratios.

Although the learned labours of all ages have failed in their attempts at the geometrical quadrature of the circle, and even of the division of the circumference into any number of equal parts at pleasure; yet our own time has furnished the solution of a problem but little less curious, and heretofore esteemed almost, if not altogether, as difficult as it; namely the division of the circle into any proposed number of parts whatever, of equal perimeter, and the areas either equal or in any proportion to each other. The solution of this seeming paradox was first published by Dr Hutton, in his quarto volume of Tracts, in 1786. That curious solution was, in substance, as follows:

Divide the diameter AB (fig. 132 pl. 17) of the given circle into as many equal parts as the circle itself is to be divided into, at the points C, D, E, &c. Then, on the lines AC, AD, AE, &c, as diameters, describe semicircles on one side of the diameter AB, and also on the lines BE, BD, BC, &c, on the other side of that diameter; then will these semicircles divide the whole given circle in the manner proposed, viz, into parts which are all equal to each other, both in area and in perimeter.

For, the several diameters of the dividing semicircle being in arithmetical progression, and the diameters of circles being in the same proportion as their circumferences, these will also be in arithmetical progression. But, in such a progression, the sum of the extremes being equal to the sum of each pair of terms that are equally distant from them; therefore the sum of the circumferences on AC and CB, is equal to the sum of those on AD and DB; and to the sum of those on AE and EB, &c, and each sum equal to the semicircumference of the given circle on the whole diameter AB. Therefore all the parts have equal perimeters; and each perimeter is equal to the whole circumference of the first given circle. Which satisfies one of the conditions in the problem.

Again, the same diameters being in proportion to each other as the numbers, 1, 2, 3, 4, &c, and the areas of circles being as the squares of their diameters, the semicircles will be as the square numbers 1, 4, 9, 16, &c, and consequently the differences between all the adjacent semicircles are as the terms of the arithmetical progression, 1, 3, 5, 7, &c: and here again the sums of the extremes and of every two equidistant means, make up the several equal parts of the circle. Which is the other condition of the problem.

But this subject admits of a still more geometrical form, and is capable of being rendered very general and extensive, and is moreover very fruitful in curious consequences. For first, in whatever ratio the whole diameter is divided, whether into equal or unequal parts, and whatever be the number of the parts, the perimeters of the parts will always be equal. For since the circumferences of circles are in the same proportion as their diameters, and because AB (fig. 123 pl. 17) and AD + DB, and AC + CB are all equal, therefore the semicircumferences c, and $b + d$, and $a + e$ are all equal, and constantly the same, whatever be the ratio of the parts, AD, DC, CB, of the diameter. We shall presently find too that the spaces TV, RS, and PQ, will be universally as the same parts, AD, DC, CB, of the diameter.

The semicircles having been described as before mentioned, erect CE perpendicular to AB, and join BE. Then will the circle on the diameter BE, be equal to the space PQ. For, join AE. Now the space P is = semicircle on

AB − semicircle on AC: but the semicir. on AB = semicir. on AE + semicir. on BE, and the semicir. on AC = semicir. on AE − semicir. on CE, therefore semic. AB − semic. AC = semic. BE + semic. CE, that is, the space P is = semic. BE + semic. CE; to each of these add the space Q, or the semicircle on BC, then P + Q = semic. BE + semic. CE + semic. BC, that is, P + Q = double the semic. BE, or = the whole circle on BE.

In like manner, the two spaces PQ and RS together, or the whole space PQRS, is equal to the circle on the diameter BF. And therefore the space RS alone is equal to the difference, or the circle on BF minus the circle on BE.

But, circles being as the squares of their diameters, BE^2, BF^2, and these again being as the parts or lines BC, BD, therefore the spaces PQ, PQRS, RS, TV, are respectively as the lines BC, BD, CD, AD. And if BC be equal to CD, then will PQ be equal to RS, as in the first or simplest case.

Hence, to find a circle equal to the space RS, where the points D and C are taken at random: From either end of the diameter, as A, take AG equal to DC, erect GH perpendicular to AB, and join AH; then the circle on AH will be equal to the space RS. For, the space PQ is to the space RS, as BC is to CD or AG, that is as BE^2 to AH^2, the squares of the diameters, or as the circle on BE to the circle on AH. But the circle on BE is equal to the space PQ; therefore the circle on AH is equal to the space RS.

Hence, to divide a circle in this manner, into any proposed number of parts, that shall be in any ratio to one another: Divide the diameter into as many parts, at the points D, C, &c, and in the same ratios as those proposed; then on the several distances of these points from the two ends A and B, as diameters, describe the alternate semicircles on the different sides of the whole diameter AB; and they will divide the whole circle in the manner proposed. That is, the spaces TV, RS, PQ, will be as the lines AD, DC, CB.

But these properties are not confined to the circle alone. They are to be found also in the ellipse, as the genus of which the circle is only a species. For if the annexed figure be an ellipse described on the axis AB (fig. 131 pl. 17) the area of which is, in like manner, divided by similar semiellipses, described on AD, AC, BC, BD, as axes, all the semiperimeters *f*, *ae*, *bd*, *c*, will be equal to one another, for the same reason as before in the circle, namely, because the peripheries of similar ellipses are in the same proportion as their diameters. And the same property would still hold good, if AB were any other diameter of the ellipse, instead of the axis; describing on the parts of it semiellipses which shall be similar to those into which the diameter AB divides the given ellipse.

And further, if a circle be described about the ellipse, on the diameter AB, and lines be drawn similar to those in the second figure; then, by a process the very same as before in the circle, substituting only semiellipse for semicircle, it is found that the space

PQ = the similar ellipse on the diameter BE,
PQRS = the similar ellipse on the diameter BF,
RS = the similar ellipse on the diameter AH, or to the difference of the ellipses on BF and BE; also the elliptic spaces PQ, PQRS, RS, TV, are respectively as the lines BC, BD, DC, AD; being the same ratios as the circular spaces. And hence an ellipse is divided into any number of parts, in any assigned ratios, after the same manner as the circle is divided, namely, dividing the axis, or any diameter, in the same manner, and on the parts of it describing similar semiellipses.

To divide a Circle by other Concentric Circles.

This is another variety of the same problem, and which has been originally proposed in this manner. A certain number of persons purchased a grinding stone among them, in equal shares; and they agreed that they should, one after another, grind down an equal portion of it, or an equal part of the area of its plane surface.

Let ADBE (pl. 17 fig. 135) represent the flat or plane circular surface of the grindstone; and suppose the number of persons to be 5; then the problem is, to divide this circle into 5 equal portions, by means of smaller circles, concentric with the given circle, or having the same centre C. This question was proposed in the Ladies' Diary for the year 1709, and the solution of it, in the following manner, was first given by Dr Hutton.

Divide the given radius AC into the given number (5) of equal parts; at the points of division erect perpendiculars, to meet the semicircle described on the diameter AC, in the points *e*, *f*, *g*, *h*; then through these points, from the centre C, describe as many circles; and they will divide the given circle in the manner required.

For the areas of the several included surfaces are proportional to the squares of their radii, C*e*, C*f*, C*g*, C*h*; but the squares of these radii, by the principles of circles, are as the versed sines C*d*, C*c*, C*b*, C*a*; therefore the circles are as these versed sines; but these versed sines, by the construction, are in arithmetical progression; therefore the areas of the circles are also in arithmetical progression; consequently their differences, or the intervals between their circumferences, are all equal.

PROBLEM LXV.

Of various other Circular Spaces absolutely Squarable.

1st. Let there be two concentric circles, through which is drawn the line *b*B (fig. 103 pl. 12), a tangent or secant to the interior circle. Draw CA and CB, forming an angle ACD, and make the arc DF in proportion to the arc DA, as the square of CD is to the difference of the squares of CB and CD: if CE be then drawn, we shall have the mixtilineal space ABEF equal to the rectilineal triangle ACB.

It is evident that, to render the position of CE determinable by common geometry, the ratio between the arcs AD and DF must be that of certain numbers, as 1 to 1, 1 to 2, 1 to 3, &c, or 2 to 1, 2 to 3, &c. Consequently, the difference of the squares of the radii of the two circles, must be to the square of the less, as 1 to 1, or 2 to 1, or 3 to 1, &c. The sectors of the different circles being then in the compound ratio of the squares of their radii and of their amplitudes, we shall have the sector BCE equal to ACF; if the common sector DCF therefore be taken away, and the space ADB be added to both, the rectilineal triangle ACB will be equal to the space AFEB.

2d. Let there be any sector, as ACBGA (fig. 104), of which AB is the chord. In a double, or quadruple, or octuple circle, take a sector *acbga*, the angle of which shall be the half, or the fourth, or the eighth part of the angle ACB, which it is possible to do with the rule and compasses; let this second sector be disposed as seen in the figure, that is to say in such a manner, that the arc *agb* shall stand on the chord AB. We shall then have the space A*agb*BGA, equal to the rectilineal figure ECF*c*, minus the two triangles A*a*E and *b*BF.

This is almost evident; for, by the above construction, the sector ACBG is equal to *acbg*; if the part therefore which is common be taken away, there will be an equality between what remains on the one hand, viz, the kind of lunule AGB*bga*, plus the two triangles A*a*E, and B*b*F, and what remains on the other, or the rectilineal figure E*c*FC: this kind of lunule therefore is equal to the above rectilineal figure, diminished by these two triangles.

3d. If two equal circles cut each other in A and B (fig. 105), and if any line AC be drawn, intersecting the interior arc in E, and the exterior in C, it is evident that the arc BE will be equal to the arc BC; and consequently the segment EB will be equal to the segment BC. Hence it follows, that the triangle formed by the two arcs EB and BC and the straight line EC, will be equal to the rectilineal triangle EBC. Lastly, that if AD be a tangent in A to the arc AEB, the mixtilineal figure AEBCDA will be equal to the rectilineal triangle ADB.

4th. If two equal circles touch each other in C (fig. 106), and if a third equal circle be described through the point of contact; the curvilineal space AFCEDBA will be equal to the rectilineal quadrilateral ABDC. For, if CB be drawn a tangent to the first two circles, the space comprehended by the arcs CFA and AB and the straight line CB, is equal to the rectilineal triangle CAB, as has been shown already. The case is the same with the mixtilineal space CEDB in regard to the triangle CDB: therefore, &c.

5th. The above remark was made by M. Lambert, in the Acta Helvetica, vol. iii. But other spaces of the same form may be found equal to rectilineal figures, though bounded by circular arcs, two of which only are equal.

Let ABCD (fig. 107) be a circle, from which it is required to cut off, by two other circular arcs, a space of the above kind absolutely squarable. On an indefinite right line make the parts CE, EF, FH, each equal to the side of the square inscribed in the given circle; and let the third part FH be divided into two equal parts in G: on the extremity of CE raise the perpendicular EI, and let it be intersected in I, by a circle described from G as a centre with the radius GC. Draw CI, and make CK equal to it; lastly, on FG describe a semicircle, cutting, in L, the line KL perpendicular to FG; draw HL, and in the given circle make the chords AB and AD equal to it. If with a radius equal to CE, there be then described arcs, passing through the points A and B, A and D, with their convexity turned towards C; we shall have the space bounded by the arcs AB, AD and BCD, equal to the recti-lineal space formed by the chords AB, AD, and the four chords DM, MC, CN and NB, of the four equal portions of the arc BCD.

But, as enough has been said on this subject, we shall only add one reflection, which is, that these quadratures ought not to be considered as real quadratures of a curvilineal space. All the marvellous in these operations, as M. de Fontenelle has very properly remarked, consists in a kind of geometric legerdemain, by means of which as much is dexterously added on the one hand, to a rectilineal space, as is taken from it on the other. It was not in this manner that Archimedes first squared the parabola, and in which modern geometricians have given the quadrature of so many other curves. All these things however appeared to us sufficiently curious to entitle them to a place in a work of this nature.

PROBLEM LXVI.

Of the measure of the Ellipse or Geometrical Oval, and of its parts.

It may be easily demonstrated, that the ellipse (fig. 109 pl. 13) is to the rectangle of its axes AB and DE, as the circle is to the rectangle of its axes, or to the square of its diameter AB, since each axis is equal to the diameter.

Thus, as the circle is $\frac{11}{14}$ nearly of the square of its diameter, the ellipse is also $\frac{11}{14}$ of the rectangle of its axes.

Nothing then is necessary, but to multiply the rectangle of the axes of the given ellipse by 11, and to divide the product by 14; the quotient will give the area.

We shall here add, that each segment or sector of the ellipsis is always in a given ratio to the sector or segment of a circle, as is easy to be determined. Let the elliptical sector FCG, (fig. 110) for example, or the segment FBG, be given: on the axis AB describe a circle from the centre C; and if GF be continued to D and E, we shall have the elliptical sector FCGB to the circular sector DCEB, as FG to DE, or as the less axis of the ellipsis is to the greater: the elliptical segment BFG will also be to the circular segment DBE, as FG to DE, or as the less axis of the ellipsis to the greater.

Let there be likewise, in an ellipsis, any segment whatever, as *nop*. On the axes let fall two perpendiculars from *n* and *p*, and continue them till they meet the circle in N and P; if NP be then drawn, we shall have the segment *nop* to the circular segment NOP, in the same ratio as the less axis is to the greater. From this is deduced the solution of the following problem.

PROBLEM LXVII.

To divide the sector of an ellipsis into two equal parts.

Let it be required, for example, to divide the elliptical sector DCB (fig. 111) into two equal parts, by a line as CG.

On the diameter AB describe a circle; and having drawn DI perpendicular to AB, continue it to E, and draw EC, which will give the circular sector ECB; divide the arc EB into two equal parts in F, and draw FH perpendicular to the axis AB; then from the centre C, to the point G, where that perpendicular cuts the ellipsis, draw the line GC: the elliptical sector BCG will be equal to GCD, as the circular sector BCF is to FCE.

The case would be the same if the sector were equal to the 4th part of an ellipsis, or any higher part; and also if the sector were comprehended between any two semidiameters of the ellipsis, as DC and *d*C.

In this case, from the points D and *d*, let fall on the axis the perpendiculars DI and *di*, which when continued will cut the semicircle AEB in E and *e*; divide the arc E*e* into two equal parts in *f*, and draw *fh* perpendicular to AB, cutting the ellipsis in *g*: the line C*g* will divide the sector DC*d* into two equal parts.

PROBLEM LXVIII.

A carpenter has a triangular piece of timber; and, wishing to make the most of it, is desirous to know by what means he can cut from it the greatest right-angle quadrangular table possible: In what manner must he proceed?

Let the given triangular piece of timber be ABC (fig. 112 pl. 13). Divide the two sides AB, BC into two equal parts, in F and G, and draw FG; then from the points F and G draw FH and GI perpendicular to the base: the rectangle FI, will be the greatest possible that can be inscribed in the triangle, and will be exactly the half of it.

If the triangle be right-angled at A (fig. 113) the question may be solved, in two different ways, by which there may be obtained the two rectangular tables F*i* and FI, which will each be the greatest inscriptible in the given triangle, and both equal.

When the triangle has all its angles acute, the solution will be different according to the side assumed as base. There will consequently be three, and each will give a table more or less elongated, and always of the same area, otherwise the greatest would exclusively solve the problem: such are the rectangles FI, GL, and K*m* (fig. 114).

But the carpenter having consulted a geometrician, the latter observed that it would be most advantageous to convert this piece of timber into an oval table: in what manner then must he proceed to trace out on it the greatest oval possible.

Let the given triangular piece of wood, as before, be ABC (fig. 115). First divide each side into two equal parts in F, D, and E; these three points will be the points of contact where the ellipsis touches the sides of the triangle; if the lines AE, CF and BD be then drawn, intersecting each other in G, the point G will be the centre of the ellipsis.

Then make GL equal to GE, and through G draw GO, parallel to BC, and through the point D draw DQ parallel to AE; then take GP a mean proportional between GQ and GO: if the triangle BAC be isosceles, the lines GL and GP will be the semi-axes of the ellipsis; and we have already shown in what manner an ellipsis may be described when the two axes are given.

But if the angle LGP be acute or obtuse, the ellipsis may be traced out at once by means of an instrument, described in Prob. XXII; for it is of little importance whether the angle of the two given diameters be a right angle or not. This method will always be equally successful; with this only difference, that when the above angle is not a right angle, the portions of the ellipsis, described in the two adjacent angles, LGP and LGR, will not be equal and similar.

The two axes may be determined also directly: the method may be found in books on conic sections, and to these we must refer, as the nature of this work will not admit of entering deeply into the subject.

PROBLEM LXIX.

The points B *and* C (fig. 128 pl. 16) *are the adjutors of two basons in a garden, and* A *is the point where a conduit is introduced, and to be divided into two parts, in order to supply* B *and* C *with water. Where must the point of separation be, that the sum of the three conduits* AD, DB *and* DC, *and consequently the expence in pipes, shall be the least possible?*

This problem, which belongs to that branch of civil engineering that relates to the conveyance of water, when reduced to geometrical language, may be enounced as follows: In a triangle ABC to find a point, from which if three lines be drawn to the three angles, the sum of these lines shall be the least possible. Now it is evident that there must be such a point, and that its position being found, the expence in pipes will be less than if the point of separation were assumed in any other place.

It would be tedious to explain the reasoning by means of which this problem is solved; and it would be impossible to employ calculation without great prolixity. We shall therefore only observe, that it may be demonstrated, that the required point D must be so situated, that the angles ADC, ADB, and BDC shall be equal to each other, and consequently each equal to 120°.

To construct this problem, on the side AC as a chord describe an arc of a circle ADC, capable of containing an angle of 120°, or equal to one third of the circle of which it forms a part; if the same thing be done on another of the sides, as BC, the intersection of these two circular arcs will determine the required point D; and it is from this point that the conduit must be divided, in order to be conveyed thence to B and to C.

Such, at least, would be the solution of the problem, if the three pipes AD, DC, DB, were all to be of the same bore. But an intelligent engineer will not make the pipes equal in size; he will be sensible that to give greater height to the jet, it will be proper that the pipes DB and DC should not together admit a greater quantity of water than the pipe AD, otherwise the water in these pipes, after coming from the pipe D, would be in a state of stagnation, and would not receive the impulse necessary to make it rise to its greatest height.

The solution of the problem, in this new case, is as follows: We shall suppose that the bore of the pipe AD, or its capacity, is exactly double that of the other two; that is to say, that the diameters are in the ratio of 10 to 7 nearly; for by these means the water will always sustain an equal pressure in the former and in the two latter. We shall suppose also, that the price of the foot of each kind of these pipes is in the same ratio, because, in economical problems of this sort, it is the ratio of the prices that ought chiefly to be considered.

These things being premised, we shall find that the point of separation of the pipes ought to be in *d*, so situated, that the angles *cd*A and B*d*A shall be equal, and of such a nature, that the sine of each shall be to radius as 10 is to 14; or more generally, as the price of the foot of the larger pipe, is to double that of the smaller. Hence it will be easy, according to this hypothesis, to determine the angle, which will be found to be 132° 56′ or near 133°.

If on the sides AB and AC then, of the triangle ABC, there be described two circular arcs, each containing an angle of 133°, their point of section will be in *d*, where the main pipe ought to be divided, to convey water to B and C, so as to incur the least possible expence in pipes.

REMARK.

By extending this problem, we may suppose that the main pipe is to convey water to three given points B, C, E (fig. 129). In that case it may be demonstrated, that if the four pipes were equal, the point of separation could not be placed more advantageously, at least for diminishing the quantity of the pipes, than in the place where the lines AE and BC intersect each other; but this perhaps would not be the most advantageous disposition for making the water to be thrown up with the greatest force.

The same observation, made in regard to the first solution of the problem, may be made here also. To give greater force to the jet, the main pipe ought to be nearly triple in size to each of the rest. Let us suppose then that the price of a foot of the former, is to that of a foot of the others, as *m* is to *n*; and in the last place, to simplify the problem, the solution of which would be otherwise exceedingly complex, we shall suppose that the lines AE and

BC cut each other at right angles: this being the case, it will be found, that the angle EFC ought to be such, that, radius being unity, the cosine of it shall be $\frac{1}{2} n \times \sqrt{4 nn - (m - 1)}$, or, what amounts to the same thing, the sine of the angle DCF must be equal to the above expression.

If we suppose then, for example, that m is to n as 5 to 3, we shall have the above expression equal to 0·71496, which is the sine of an angle of 45° 38′. If the angle DCF therefore be made equal to from 45° to 46°, the point F will be that where the principal pipe ought to be divided.

If m were to n as 2 to 1, the above expression would become equal to 0·86600, which is the sine of an angle of 60°; in this case therefore the angle DCF ought to be made equal to 60°, or each of the angles DFC and DFB equal to 30°.

It is here evident that, to render the problem susceptible of a solution, m and n must be such, that the above expression shall not be imaginary, nor greater than unity. In either of these cases there could be no solution; and this would indicate, at most, that the division ought to be made at the point A, or at as great a distance as possible from the line BC. This expression also must not be $= 0$: in that case we ought to conclude that the division should be made at the point D.

PROBLEM LXX.

Geometrical paradox of lines which always approach each other, without ever being able to meet or to coincide.

Every person, in the least acquainted with geometry, knows, that if two straight lines, in the same plane, approach each other, they will necessarily meet in a common point of intersection. We say, in the same plane; for if they were in different planes, it is evident that they might approach till a certain term, without cutting each other, and that they would then diverge from each other more and more. If we suppose, for example, two parallel and vertical planes, on one of which is drawn a horizontal line, and on the other one inclined to the horizon, it may be readily conceived that they will not be parallel, and yet they can never intersect each other, their least distance being necessarily that of the two planes. Here then we have two lines, not parallel, which never meet: but this is not the sense in which the problem is understood.

It may be demonstrated that there are many lines, and in the same plane, which continually approach each other, and which however can never meet. They are indeed not straight lines, but a curve combined with a straight line, or two curved lines together. We shall here give a few examples of these lines, which are very familiar to those who are versed in the higher geometry.

In the indefinite straight line AG, (fig. 116 pl. 13) take the equal parts AB, BC, CD, &c; and from the points B, C, D, &c, raise the perpendiculars B*b*, C*c*, D*d*, E*e*, &c, which decrease, according to a progression no term of which can become 0, though it may become indefinitely small; let these terms decrease, for example, according to the progression, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, &c: it is evident that the curve passing through the summits of the lines, decreasing according to this progression, can never meet the line G, however far continued, since its distance from that line can never become 0: it will however approach it more and more, and in such a manner, as to be nearer it than any quantity, however small. This curve, in the present case, is that so well known to geometricians under the name of the hyperbola; which has the property of being contained between the branches of two rectilineal angles, having their vertices opposed to each other, towards which it approaches more and more, without ever touching them.

If the progression, according to which these lines B*b*, C*c*, D*d*, &c, decrease, were 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c, the line passing through the points *b*, *c*, *d*, *e*, &c, would still approach more and more to the straight line AG, without ever meeting it; for whatever might be the distance of any term of this progression, it could never become = 0.

Another Example. Without the indefinite line AF, (fig. 117) assume any point P, from which draw PA perpendicular to AF, and any other lines at pleasure PB, PC, PD, &c, more and more inclined; in the continuation of which make the lines A*a*, B*b*, C*c*, &c, always equal: it is evident that the line passing through the points *a*, *b*, *c*, *d*, &c, never can meet the line AF, though it may approach it more and more, and nearer than any determinate quantity; because F*f* becomes more and more inclined. This curve is that known to geometricians by the name of the *conchoid*, and was invented by Nicomedes, a Greek geometrician, to serve for the solution of the problem respecting two mean proportionals.

A great many other examples might be found in the higher geometry; but these will be sufficient for our purpose.

PROBLEM LXXI.

In the island of Delos, a temple consecrated to Geometry was erected, on a circular basis, (fig. 118 pl. 14), *and covered by a hemispherical dome, having four windows in its circumference, with a circular aperture at the top, so combined, that the remainder of the hemispherical surface of the dome was equal to a rectilineal figure; and in the cylindric part of the temple was a door, absolutely quadrable, or equal to a rectilineal space. What geometrical means did the architect employ in the construction of this monument?*

Every person, acquainted with the principles of geometry, knows that the measure of a hemispherical surface depends on that of the circle, which is equal to the surface of a cylinder having the same base and the same altitude.

The ingenuity of this construction then was, 1st. To have cut from the dome, by the apertures above mentioned, spherical portions of such a nature, that the remainder should be equal to a figure purely rectilineal. 2d. To have described in the cylindric part, or circular wall of the temple, another figure which was squarable. The method that might have been employed is as follows:

Let us first suppose a 4th part of the hemispherical dome, having for its base the quadrant ACB (fig. 119). Take the arc BD, equal to $\frac{1}{4}$ of the arc AB, as the breadth of the arc that ought to separate the windows; and draw AD the chord of the remainder. Now let SCE be any section whatever, through the axis of the dome SC, and let its intersection with AD be F; make CE, CF, CG continually proportional; in the axis CS make the line CH equal to EG, and draw HI parallel to CE, which will intersect the quadrant SE in I: then will I be one of the points of the window required; and the series of points I, determined in this manner, will give the contour of that window, the surface of which will be equal to double the segment AED, while the spherical portion SAIDS will be equal to double the rectilineal triangle CAD.

The whole surface of this 4th part of the dome will be equal then to double this triangle, plus the spherical sector SDB, which is equal to double the circular sector CDB, or to the 4th of the spherical sector SAEB; if from this sector therefore, there be cut off the 4th part SLM by a plane parallel to its base, and distant from the vertex S by the 4th part of the radius SC, the remainder of this hemispherical quadrant, that is to say the surface AIDBMLA, will be equal to double the rectilineal triangle CAD. If the other quadrants of the hemispherical dome be then made similar to the present one, the whole dome, the apertures deducted, will be equal to 8 times the triangle ACD.

In regard to the aperture to be made in the circular wall of the temple, and which must be equal to a rectilineal space, nothing is easier, though it be a part of a cylindric surface. Let ABDEF (fig. 120) represent one half of this surface; assume, as the breadth of the door to be formed, the chord GH, parallel to the diameter AD; make GI and HK, which are perpendicular to the base, of such a size, that the door may have that proportion which good taste and the character of the work require; if through the points I, K, and the line AD, a plane be then made to pass, which by its intersection with the cylindric surface will determine the curve ILK, we shall have the cylindric aperture GBHKI, a little arched at the top, which will be to the rectangle of CB by GH, as the sine of the angle LCB is to the sine of half the right angle. The problem of the Greek geometrician therefore is solved.

This problem might be varied a great many ways. During my dreary residence, in 1758, at a post in Canada, I amused myself with these variations, and I resolved the problem by making the whole of the surface of the temple absolutely quadrable. I left only one aperture in the dome, viz, a hole at the top, like that of the Pantheon at Rome, and I made the four windows in the cylindric part of the temple, &c. All this however will be easy to any one versed in geometry.

REMARKS.

1st. This problem is nearly the same as that proposed by Viviani, in 1692, under the title of Ænigma Geometricum, which was easily solved by Leibnitz, Bernouilli, and the Marquis de l'Hôpital. An account of it may be seen in my History of the Mathematics, Vol. II. Book 1. Viviani's solution is ingenious and elegant; but as the dome, according to this solution, would not be susceptible of construction, because it would bear upon four points, which in architecture is absurd, we have made some changes in the enunciation, by adding the circular aperture at the top. By these means the dome will bear upon parts that have some solidity, each window being separated from the other by an arc which forms a 6th part of the whole circumference.

2d. Father Guido-Grandi has remarked, that if a polygon, for example the triangle ABC, (fig. 121) be inscribed in the circular base of a cone, and if on each side of this polygon a plane be raised perpendicular to the base, the portion of the conical surface, cut off towards the axis, is equal to a rectilineal space. For it may be easily demonstrated, that this surface is to that of the rectilineal polygon ABC, which corresponds to it perpendicularly below, as the surface of the cone, is to the circle of its base; that is, as the inclined side of the cone SD, is to ED the radius of that base.

The portions also of the cone cut off by the above planes, towards the base, are evidently in the same ratio with the segments of the circle on which they rest. In fact, whatever figure be inscribed in the base, if we conceive a right cylindric surface raised from the circumference of the figure, it will cut off from the conical surface a portion which will be to it in the same ratio.

This Italian geometrician, who was of the order of the Camaldules, thought proper to give to this conical portion absolutely quadrable, the name of Velum Camaldulense. In like manner, a Franciscan took it into his head to construct a sun-dial on a body which resembled a sandal, and to print a description of it, under the title of Sandalion Gnomonicum.

PROBLEM LXXII.

If each of the sides of any irregular polygon whatever, as ABCDEA, (fig. 122 pl. 14), *be divided into two equal parts, as in* a, b, c, d, e; *and if the points of division in the contiguous sides be joined; the result will be a new polygon* abcdea: *if the same operation be performed on this poly-*

gon; then on the one resulting from it; and so on ad infinitum; it is required to find the point where these divisions will terminate.

This problem, impossible to be resolved perhaps by considerations purely geometrical, is susceptible of a very simple solution, deduced from another consideration, and which shall be given in the next volume.

In the mean time our readers may exercise their ingenuity upon it, as we shall only add, that it was proposed in 1750 by M. D———, who said he had it from M. Buffon.

A COLLECTION

Of Various Problems, both Arithmetical and Geometrical; the solution of which is proposed by way of exercise to mathematical readers.

THOSE who study mathematics cannot begin too early to exercise their talents with the solution of the problems presented by that science; for it is by such exercise that the inventive faculty is called forth and strengthened. We have therefore thought it our duty to subjoin to this part of the Mathematical Recreations a selection of problems proper for exercising and amusing young mathematicians. They are of different degrees of difficulty, that they may be suited to the different capacities of those who read this work. Some curious theorems have been inserted among them; and, as the demonstration of these is required, they may serve also to exercise their ingenuity.

It may be here observed, that, as the most of these problems are far from being difficult if the resources of algebraic calculation be employed, it is therefore proposed that the solutions of them should be found by means of pure geometry; as it is well known that algebraic analysis gives, for the most part, complex solutions, while those which arise from analysis purely geometrical are far more simple and elegant.

ARITHMETICAL AND GEOMETRICAL PROBLEMS AND THEOREMS.

PROBLEM I. **In a right-angled triangle, given the base, the sum or difference of the other two sides, and the area, to determine the triangle?**

PROB. II. Given the base, the ratio of the other two sides, and the area, to determine the triangle.

PROB. III. The base, the angle comprehended by the two other sides, and the area being given, to determine the triangle.

PROB. IV. Three lines being given in position, on a plane, to draw another line through them, which shall be cut by them into two parts, in a given ratio.

PROB. V. Four lines being given in position on a plane, to draw another line through them, which shall be cut into three parts, in a given ratio.

PROB. VI. What is the probability of throwing an ace, or any one of the faces of a die, in three throws; that is either at the first, second or third throw?

PROB. VII. At the game of Piquet, A is first in hand, and has no ace; what probability is there that he will take in from the pack either one, or two, or three, or four aces?

PROB. VIII. What is the probability of throwing one ace, and no more, in four successive throws?

PROB. IX. In a lottery, where the number of blanks is to that of the prizes as 39 to 1, as was the case in the year 1720, how many tickets must be purchased that the buyer may have an equal chance for one or more prizes?

PROB. X. If a man has in his hand a certain number of pieces of money, as for example 12, how much may be betted to 1 that in tossing them all up at once, or separately, there shall be as many heads as tails?

PROB. XI. Four lines being given of such a nature, that any three of them are together greater than the fourth, to construct of them a quadrilateral figure inscriptible in a circle, or which can be circumscribed about it?

THEOREM I. If from the three angles of any right lined triangle, three lines be drawn perpendicular to the opposite sides, they will all cut each other in the same point.

THEOR. II. If lines be drawn from these angles, dividing each of them into two equal parts, or cutting the opposite sides into two equal parts, these three lines will all pass through the same point.

PROB. XII. A trapezium being given, to divide it into two equal parts, or in any given ratio, by a line passing through a given point, either in one of the sides, or within the trapezium, or without it.

PROB. XIII. In a given circle to inscribe an isosceles triangle of a given magnitude.—It is evident that this triangle must be less than the equilateral triangle inscribed in the given circle; for the latter is the greatest of all those that can be inscribed in it.

PROB. XIV. To circumscribe about a given circle an isosceles triangle of a given magnitude.—This triangle must be greater than the circumscribed equilateral triangle; since the latter is the least of all those that can be circumscribed.

PROB. XV. In an isosceles triangle to describe three circles, each of which shall touch two sides of the triangle, and which all three shall touch each other.

PROB. XVI. To do the same thing in a scalene triangle.

PROB. XVII. What is the value of this analytical expression, $\sqrt{2\sqrt{2\sqrt{2}}}$ &c, in infinitum?—The answer is 2; but a demonstration is required. In like manner the value of $\sqrt{3\sqrt{3\sqrt{3}}}$ &c, in infinitum, is 3; and so of any other number.

PROB. XVIII. In a pyramid, of four triangular faces, if the sides of these four triangles be given; required the angles formed by the faces of this pyramid, the perpendicular let fall from any of the angles on the base, and the solidity of the pyramid.

PROB. XIX. To cut a given trapezium into four equal parts, by lines intersecting each other at right angles.

PROB. XX. A gentleman has an irregular quadrangular piece of ground, from which he is desirous, for the purpose of making a parterre, to cut the largest oblong possible, with its angles touching the sides of the quadrilateral: how is this to be done?

PROB. XXI. Given the area of a right-angled triangle, and the sum of the three sides, to determine the triangle.

PROB. XXII. If from a pack, consisting of 52 cards, 13 of each suit, 5 cards be dealt to one person; what is the chance that two of them shall be trumps, or of any suit that is proposed?

PROB. XXIII. About a given circle to circumscribe a triangle, of a given perimeter; provided this perimeter be greater than that of the equilateral triangle circumscribed.

PROB. XXIV. In a triangle, not equilateral, to find a point, from which, if three perpendiculars be drawn to the three sides, they shall be together equal to a given line.—We have excluded the equilateral triangle, because it may be easily demonstrated, that from whatever point, within such a triangle, perpendiculars are let fall on the sides, their sum will be always the same.

The case is the same in regard to every regular polygon; and even those that are irregular, provided the sides are equal.

PROB. XXV. In a given circle to inscribe an isosceles triangle, or to circumscribe about it a triangle of a given perimeter.—This problem not being always possible, as may be easily seen, it is required to assign its limitations.

PROB. XXVI. In a given circle to inscribe, or to circumscribe about it, any triangle whatever, of a determinate perimeter.

PROB. XXVII. In a given quadrilateral to inscribe an ellipsis; that is, to describe in it an ellipsis which shall touch its four sides.

PROB. XXVIII. A jeweller has a valuable plate of agate, in the form of an irregular trapezium, and is desirous to cut from it the largest oval possible for the lid of a snuffbox: in what manner must he proceed?—It is evident that this problem expressed geometrically is as follows: In a given quadrilateral to inscribe the largest ellipse possible: a problem which is certainly not easy. It is proper to inform those who may be disposed to try it, that it requires a profound knowledge of analysis. The following also might be proposed: About a given quadrilateral to circumscribe the least ellipsis possible.

PROB. XXIX. A point and a straight line being given, in what line will be found the centres of all the circles passing through the given point, and touching the given line.

PROB. XXX. Required the same thing in regard to all the circles that touch a given circle and a given straight line.—This straight line may be without the given circle; or it may touch it, or intersect it.

PROB. XXXI. Any two circles being given, in what line will be found the centres of all the circles that touch the given circles; whether the touching circle comprehends them both within it, or touches the one without and the other within?

PROB. XXXII. The base of a triangle, the sum of the two other sides, and the line drawn from the vertex to the middle of the base, being given; to determine the triangle.

PROB. XXXIII. Given the three lines, drawn from the angles of a triangle to the middle of each of the opposite sides; to determine the triangle.

PROB. XXXIV. Given the base of a triangle, and the sum and the difference of the squares of the sides, to determine the triangle.—This problem is susceptible of a very simple and very elegant construction; for the vertex of this triangle is in the circumference of a certain circle, and is also in a certain straight line.

PROB. XXXV. Given the three lines drawn from the angles of a triangle to the opposite sides, dividing each of these angles into two equal parts; to determine the triangle.

PROB. XXXVI. Any number of points being given, to draw a straight line among them in such a manner, that if a perpendicular be let fall on it from each of these points, the sum of the perpendiculars on the one side, shall be equal to the sum of those on the other side.

PROB. XXXVII. The same supposition being made, it is required that the sum of the squares of the perpendiculars drawn on the one side, shall be equal to the sum of the squares of those on the other; or that the sum of these perpendiculars, raised to any power whatever *n*, shall be on both sides equal.

PROB. XXXVIII. In any trapezium, given the four sides and the area, to determine the trapezium.

PROB. XXXIX. An angle being given, to find a point from which if two perpendiculars be let fall on its sides, the quadrilateral formed by them and the sides of the angle, shall be equal to a given square.

PROB. XL. As there are an infinite number of points which will answer the problem, it is proposed to find the line traced out by them, or the curve which they form.

PROB. XLI. To find four numbers in arithmetical progression, to which if four given numbers, such as 2, 4, 8, 17, be added, their sums shall be in geometrical progression.

PROB. XLII. Two couriers, A and B, set out at the same time; A from Paris for Orleans, the distance between which is 60 miles, and B from Orleans for Paris; and they travel at such a rate that A reaches Orleans 4 hours after meeting B, and B reaches Paris 6 hours after meeting A: how many miles per hour did each travel?

PROB. XLIII. A certain sum, placed out at interest, amounted at the end of a year to 110*l*. and at the end of 18 months to 112*l*. what was the sum, and at what rate of interest was it lent?

PROB. XLIV. Two bills of exchange, one of 1200l. payable in 6 months, and the other of 2000l. payable in 9 months, were discounted at the same time, and at the same rate of interest, for 120l. at what rate of interest were they discounted?

PROB. XLV. How many ways can 100l. be paid by guineas, at 21 shillings, and pistoles at 17 shillings each?

PROB. XLVI. An angle and a point within it being given, to draw through that point a straight line intersecting the two sides of the angle, in such a manner, that the rectangle of their segments towards the vertex shall be equal to a given square.—This given square must not be less than a certain square, which gives rise to the following problem.

PROB. XLVII. The same supposition being made as in the preceding case, required the position of the line passing through the given point, when the rectangle of the sides of the angle cut off towards the vertex is the least possible.

PROB. XLVIII. Three lines being given in position, to find a point from which the three perpendiculars drawn to these lines shall be in a given ratio.——We shall here observe that this problem is susceptible of a very simple and very elegant solution, without calculation.

PROB. XLIX. Given two circles in a given ratio, as of 1 to 2, for example, and which cut each other, but in such a manner as not to form a quadrable lunule; it is proposed to draw through these circles a line parallel to that which joins the points of intersection, so that the part of the lunule cut off above may be equal to a rectilineal space.

PROB. L. The same supposition being made, it is proposed to cut the two circular arcs by a third, which shall be of such a nature that the concavo-convex triangle, formed by these three arcs, shall be equal to a rectilineal space, if possible.

PROB. LI. Three persons have together 100l; and it is known that 9 times the money of the first, plus 15 times that of the second, plus 20 times that of the third, is equal to 1500l. How much money has each?—It may be here proper to observe, that this problem, as well as the 45th, 52d, 57th, and 58th, is susceptible of several solutions; and to solve them completely it will be necessary to find all the different answers, and to show that there can be no more; for by repeated trials it would not be difficult to find some of them.

PROB. LII. A farmer bought 100 calves, sheep, and pigs, for the sum of 100l. at the rate of 3l. 10s. for the calves, 1l. 6s. 8d. for the sheep, and 10s. for the pigs: How many of each kind did he purchase?

PROB. LIII. Three merchants enter into partnership, and agree to advance each 10000l. towards a certain adventure; two of them paid down the money, but the third advanced only the half of his share, that is 5000l. the adventure having failed, they lost not only their capital but 50 per cent more: What must each contribute to make good the loss?

PROB. LIV. In a rectilineal triangle, given the base, the rectangle of the other two sides, and the including angle, to determine and construct the triangle.

PROB. LV. An arc of a circle being given, to divide it into two parts, the sines of which shall be in a given ratio.

PROB. LVI. If a person draws 4 cards from a pack, containing 32, what probability is there, or how much may be betted to 1, that among these 4 cards there will be one of each colour?

PROB. LVII. It is required to divide 24 into three such parts, that if the first be multiplied by 36, the second by 24, and the third by 8, the sum of those products may be 516?

PROB. LVIII. How many ways may four sorts of wine, the prices of which are 16, 10, 8 and 6d. per quart, be mixed, so as to make 100 quarts in all, worth 12d. per quart?

PROB. LIX. To find a number of such a nature, that if 12 and 25 be successively added to it, the sums shall be square numbers.

PROB. LX. To find three numbers, the squares of which shall be in arithmetical progression.

PROB. LXI. Any number of points being given, to find another, from which if straight lines be drawn to all the rest, the sum of these lines shall be equal to a given line.

PROB. LXII. The same supposition being made as before, the sum of the squares of the lines drawn from the required point must be equal to a given square.

It is very singular that the last problem is susceptible of a construction much easier than the preceding. We shall here observe, merely for the purpose of exciting the curiosity of the geometrical reader, that in the latter the required point, and all those that solve the problem, for there are a great many which do so, are situated in the circumference of a certain circle; and it is very remarkable that the centre of this circle is the centre of gravity of the given points, supposing each of them to be charged with the same weight.

It may be observed also, that if it were required that the square of one of the lines drawn, plus the double of the second, plus the triple of the third, &c, should make the same sum, it would be necessary to suppose the first point loaded with a single weight, the second with a double weight, the third with a triple one, &c, and their centre of gravity would still be the centre of the required circle.

The solution of this problem was not unknown to the ancient geometricians. It was one of those of the Loca plana of Apollonius; and this may serve to give us a more favourable idea of their analysis than is generally entertained.

APPENDIX.

IN the last paragraph of prob. 2, page 103, a principle is advanced, which requires some modification or limitation, viz, that "if an indeterminate doublet be proposed, it is evident that the probability is 6 times as great as when an assigned doublet is proposed; &c." For this property can only be true with 2 dice and 3 dice, but not with 4, or 5 or 6. The probability of an assigned doublet with 4 dice, is $\frac{171}{1296}$, as determined in the preceding paragraph; if this be multiplied by 6, it gives $\frac{1026}{1296}$; and if to this we add the probability of a different face coming up with each die, which is $\frac{6}{6} \times \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6}$, or $\frac{360}{1296}$, it gives $\frac{1386}{1296}$: bein 90 chances more than there are in all the 4 dice: which is impossible.

The probability of an assigned doublet with 4 dice, viz, $\frac{171}{1296}$, includes the probability of some other doublet; for we may throw aces, and also fours, or any other doublet, at the same throw; which cannot happen with two, or three, dice. So that the multiplier 6 will answer to the probability of an indeterminate doublet, with two or three dice, but not with more.

In all such cases indeed, the safest, as well as the easiest way, will be to find the chances, or the probability, of the contrary or reverse problem, namely, of not throwing doublets; and then subtracting it from the whole number of chances, or certainty, to give the chances or probability for doublets.

Thus, with two dice, A and B. For an assigned doublet, as suppose aces: each die having 6 faces, there is but one chance out of 6 that A shall come up an ace, and also one chance out of 6 that B shall come up an ace; there is therefore only one chance out of the 6 times 6, or 36, that they shall both be aces; that is, there is only one way, out of all the 36 varieties, by which the assigned doublet, aces, can come up, but 35 for the contrary. Therefore the probability of throwing it, is $\frac{1}{36}$, and that of missing it, $\frac{35}{36}$; the odds being 35 to 1 against it.—But, for an indeterminate doublet, or any doublet, whatever it may be, it is plain that, whatever face of A may come up, we have only to find the chances for B missing that face, to have the chances against the doublet. Now there being 5 faces on B different from the face of A that may have come up, and only one that is the same as it, there are therefore 5 chances out of 6 for missing the face of A; consequently the probability of missing A's face, or of missing a doublet, is $\frac{5}{6}$; which being taken from all the chances, $\frac{6}{6}$, leaves $\frac{1}{6}$ or $\frac{6}{36}$, for the chance of throwing some doublet or other: which is therefore 6 times ($\frac{1}{6}$) the chance of throwing the assigned doublet.—Or thus: the die A may come up 6 different ways; and the die B may come up with 5 faces all different from that of A, but with only one face the same as that of A; there are therefore 6×5 or 30 chances for both different faces, and 6×1 or 6 chances for the same face, or a doublet, out of the whole 36 chances on two dice.

Again, for 3 dice, A, B, C. First, the die A may come up 6 different ways, but the die B 5 ways all different from A, and the die C 4 ways all different from A and B; therefore $6 \times 5 \times 4 = 120$ are the number of ways for all three different faces, out of $6 \times 6 \times 6$ or 216, the whole number of ways or chances with three dice; consequently $216 - 120 = 96$ is the number of ways or chances for some or an indeterminate doublet, with three dice: which therefore is equal to 6 times (16) the number of chances or ways for an assigned or proposed doublet.

Again, for 4 dice, A, B, C, D. By reasoning in the same way, it appears that $6 \times 5 \times 4 \times 3 = 360$ is the number of chances or ways for all four different faces, or missing a doublet out of the whole number, 6^4 or 1296, different varieties or ways with 4 dice; consequently 1296 —

360 = 936 is the number of chances, or $\frac{936}{1296}$ the probability of throwing some or an indeterminate doublet with 4 dice; and which is therefore not so much as 6 times ($\frac{171}{1296}$) the probability of throwing a determinate doublet with the same number (4) of dice.

Again, with 5 dice. Here, in the same manner, $6 \times 5 \times 4 \times 3 \times 2 = 720$, is the number of chances or ways of throwing all the faces of 5 dice different, or of missing a doublet, out of 6 = 7776, the whole number of varieties or chances with five dice; therefore $7776 - 720 = 7056$ is the number of chances or ways for throwing doublets; consequently $\frac{7056}{7776}$ is the probability of throwing doublets with 5 dice, and $\frac{720}{7776}$ the probability of missing them.

Again, with 6 dice, the number of chances or ways for all the six faces different, is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$, out of the whole number, $6^6 = 46656$, of chances or varieties with 6 dice. Therefore $46656 - 720 = 45936$ is the number of chances or ways for throwing doublets, with the same dice. Consequently $\frac{45936}{46656}$ is the probability of throwing a doublet with 6 dice, and $\frac{720}{46656}$ is the probability of missing all doublets, with the same number of dice.

As to the throws of 7 or more dice, it is certain that there must always be a doublet, or more; because the number of dice exceeds the number of the faces on each die.

Addition on magic squares of magic squares.

Since the remark at pp. 204, 205, 206, 207, was printed, it has been discovered that there are some inaccuracies among the numbers in the cells of the magic square there referred to, fig. 1, pl. 4, of arithmetic; for the two diagonals have not either of them the sum of their numbers equal to the general sum 2056; instead of which, the sum of the one is 2184, and of the other only 1928, the former being 128 too great, and the latter the same quantity too little. Which circumstances may, perhaps, have been occasioned by misplacing or miscopying some of the numbers, by transcribers, after the figure came out of the hands of the author.

Instead of that square therefore, we here insert another, of the same kind and extent, constructed and communicated by Isaac Dalby, esq; first professor at the Royal Military College, and is engraven on plate 5 of the arithmetic; the properties of which are the following:

1. The sum of the 16 numbers in each column or row, vertical or horizontal, and in each of the diagonals, is 2056.

2. The sum of the 4 corner numbers, of each of the 8 concentric squares, is 514, or the $\frac{1}{4}$ part of the former sum 2056.

3. Half a diagonal ascending, &c, as in art. 3, pa. 205.

4. The same with all the parallels, &c, as in 4, pa. 205.

5. If a square hole, &c, as in art. 6, p. 205.

6. If the square be cut by one of the two stronger lines, either horizontally, or vertically, through the middle, the halves may change places, and the properties of the square will remain as before.

7. This magic square is composed of four other magic squares, of 64 cells each.

8. If the 4 upper and the 4 lower horizontal rows be taken away, there will remain 2 magic squares of 64 cells each. And the like takes place when the 4 left hand, and the 4 right hand columns are detached.

9. And if we take away the 4 outward rows or columns, all round, the remainder will be a magic square of 64 cells.

10. The sum of the numbers in each of the horizontal, vertical, and diagonal ranks, in each of these lesser squares, is 1028, or just half of a rank in the great one.

11. And the properties of each of these lesser squares are similar to those of the great ones.

COLLECTION OF USEFUL TABLES.

TABLE

Of the length of the foot, or other longitudinal measure used in its stead, among the different nations, and in the principal cities in Europe.

Having frequently experienced great embarrassment, while engaged in certain researches, from not being able to obtain accurate information respecting the measures of different countries, whenever an opportunity occurred we collected with great care the proportions of these foreign measures, both ancient and modern, as compared with our own, and it is hoped our readers will consider themselves indebted to us for the following table on this subject, which there is reason to think is the fullest and most complete ever given. All the different measures are compared with the English foot, which is here supposed to be divided into 12 inches, each inch into 12 lines, and each line into 10 parts; which makes the foot to consist of 1440 of these parts. The first column in the table shows the number of these parts which each measure contains; and the second the value of it in English feet, inches, lines, and tenths of a line.

ANCIENT FEET.

	parts	feet	in.	lin.	pts.
Ancient Roman foot	1392	0	11	7	2
Greek and Ptolemaic	1453	1	0	1	3
Greek Phyleterian	1681	1	2	0	1
Foot of Archimedes or probably of Sicily and Syracuse	1051	0	8	9	1
Drusian	1570	1	1	1	0
Macedonian	1670	1	1	11	0
Egyptian	2046	1	5	0	6
Hebrew	1745	1	2	6	5
The natural *(hominis vestigium)*	1172	0	9	9	2
Arabian	1577	1	1	1	7
Babylonian	1648	1	1	8	8
	1635	1	1	7	5

MODERN FEET.

	parts	feet	in.	lin.	pts.
English	1440	1	0	0	0
Altorf	1116	0	9	3	6
Amsterdam	1335	0	11	1	5
Ancona and the ecclesiastical states	1846	1	3	4	6
Antwerp	1353	0	11	3	3
Aquileia	1624	1	1	6	4
Arles	1279	0	10	7	9
Augsburg	1399	0	11	7	9
Avignon	1279	0	10	7	9
Barcelona	1428	0	11	10	8
Basle	1360	0	11	4	0
Bergamo	2060	1	5	2	0
Berlin	1428	0	11	10	8
Besançon	1462	1	0	2	2
Bologna	1792	1	2	11	2
Bourg en Bresse and Bugey	1483	1	0	4	3
Bremen	1375	0	11	5	5
Brescia	2247	1	6	8	7
Breslaw	1620	1	1	6	0
Bruges	1079	0	8	11	9
Brussels	1299	0	10	9	9

	parts	feet	in.	lin.	pts.
Chambery and Savoy . .	1594	1	1	3	4
China { Tribunal of mathematics	1623	1	1	6	3
China { Imperial foot . .	1513	1	0	7	3
Cologne	1300	0	10	10	0
Constantinople { . .	3161	2	2	4	1
Constantinople { . .	1678	1	1	11	8
Copenhagen . . .	1511	1	0	7	1
Cracow	1684	1	2	0	4
Dantzic	1329	0	11	0	9
Delft	787	0	6	6	7
Denmark . . .	1508	1	0	6	8
Dijon	1483	1	0	4	3
Dordrecht . . .	1110	0	9	3	0
Ferrara	1896	1	3	9	6
Florence	1433	0	11	11	3
Franche-Comté . . .	1687	1	2	0	7
Frankfort on the Main . .	1343	0	11	2	3
Genoa (the palm) . .	1170	0	9	9	0
Geneva	2763	1	11	0	3
Grenoble and Dauphigny .	1611	1	1	5	1
Haerlem	1350	0	11	3	0
Halle in Saxony . . .	1407	0	11	8	7
Hamburgh . . .	1343	0	11	2	3
Heidelberg (Palatinate) . .	1300	0	10	10	0
Inspruck	1586	1	1	2	6
Leghorn	1428	0	11	10	8
Leipsic	1489	1	0	4	9
Leyden	1473	1	0	3	3
Liege	1360	0	11	4	0
Lisbon	1371	0	11	5	1
Lombardy, foot of Luitprand or Aliprand	2053	1	5	1	3
Lorraine	1377	0	11	5	7
Lubec	1343	0	11	2	3
Lucca	2787	1	11	2	7
Lyons and the Lyonnese, Fores and Baujalois	1611	1	1	5	1
Madrid	1318	0	10	11	8
Maestricht . . .	1319	0	10	11	9

	parts	feet	in.	lin.	pts.
Malta (the palm) . .	1318	0	10	11	8
Mantua (the brasso) . .	2190	1	6	3	0
Marseilles . . .	1172	0	9	9	2
Mechlin	1084	0	9	0	4
Mentz	1423	0	11	10	3
Milan { Decimal foot . .	1231	0	10	3	1
Milan { Aliprand do. . .	2053	1	5	1	3
Modena	2997	2	0	1	7
Monaco	1110	0	9	3	0
Montpellier (the pan) . .	1119	0	9	3	9
Moscow	1337	0	11	1	7
Munich	1364	0	11	4	4
Naples (the palm) . .	1240	0	10	4	0
Netherlands, *see* Maestricht.					
Nuremberg { Town foot .	1434	0	11	11	4
Nuremberg { Country do. .	1306	0	10	10	6
Padua	2024	1	5	8	4
Palermo	1076	0	8	11	6
Paris { foot . . .	1535	1	0	9	5
Paris { metre . . .	4731	3	3	5	1
Parma	2692	1	10	5	2
Pavia	2217	1	6	5	7
Prague	1424	0	11	10	4
Provence, *see* Marseilles.					
Rhinlandish foot . . .	1473	1	0	3	3
Riga	1343	0	11	2	3
Rome (the palm) . . .	1055	0	8	9	5
Rouen, as at Paris . .	1535	1	0	9	5
Savoy, *see* Chambery.					
Seville in Andalusia . .	1428	0	11	10	8
Sienna, common foot . .	1784	1	2	10	4
Stettin in Pomerania . .	1763	1	2	8	3
Stockholm . . .	1545	1	0	10	5
Strasburgh { Town foot .	1377	0	11	5	7
Strasburgh { Country do. .	1395	0	11	7	5
Toledo	1318	0	10	11	8
Turin (Piedmont) . .	2414	1	8	1	4
Trent	1729	1	2	4	9

	parts	feet	in.	lin.	pts.
Valladolid . . .	1307	0	10	10	7
Venice	1638	1	1	7	8
Verona	1609	1	1	4	9
Vicenza	1636	1	1	7	6
Vienna	1492	1	0	5	2
Vienne in Dauphigny . .	1524	1	0	8	4
Ulm	1190	0	9	11	0
Urbino	1673	1	1	11	3
Utrecht	1067	0	8	10	7
Warsaw	1684	1	2	0	4
Wesel	1110	0	9	3	0
Zurich	1410	0	11	9	0

TABLE

Of some other measures, both ancient and modern, compared with the English standard.

The ancient cubit in general was a foot and a half. The Hebrews however had three cubits:

1st. The common cubit, which was a foot and a half Hebrew measure, or 2617 of those parts of which the English foot contains 1440.

2d. The sacred and modern cubit, which was one Babylonic foot and three quarters, or 2883 or 2861 parts of the English foot.

3d. The great geometric cubit, which was 9 Hebrew feet, or 6 lesser cubits.

	Grecian feet.
The orgya of the Greeks was . . .	6
The arura	50
The plethron	100
The diplethron	200

	Roman feet.
The hexapeda of the Romans was . .	6
The decempeda	10

MEASURES OF PARIS.

	French feet.	English feet.
Toise of Paris	6	6·3959
Metre, or new measure . . .	$3\frac{23}{288}$	3·2854
The royal perch	22	23·4515
The mean perch	20	21·3195
The lesser perch, used at Paris .	18	19·1876

The acre is 100 square perches.

The are is 100 square metres.

MEASURES OF CAPACITY FOR LIQUIDS.

The Muid for liquids (Paris measure) contains 8 French cubic feet, or 16744·7071 English cubic inches.

Six French cubic inches make a poinçon, or by corruption poisson, = 7·2677 Eng. cubic inches.

		Eng. cub. inches.
2 poissons make . . .	1 demi-setier	14·5353
2 demi-setiers . . .	1 chopine	29·0707
2 chopines	1 pinte	58·1413
2 pintes	1 quarte	116·2827
4 quartes	1 grand setier	465·1308
36 grand setiers . . .	1 Muid	16744·7071

Litre . . a cubical decimetre = $1\frac{1}{20}$ pinte.

A Muid therefore is equal to 72·4871 English wine gallons, or about $1\frac{1}{7}$ hogshead.

FRENCH DRY MEASURES.

The litron contains 36 French cubic inches, or 43·606 English cubic inches.

		Eng. cub. inches.
16 Litrons make . . .	1 Boisseau	697·696
3 Boisseaux	1 Minot	2093·088
2 Minots	1 Mine	4186·176
2 Mines	1 Setier	8372·352
12 Setiers	1 Paris Muid	100468·224

Hence the French Muid for things dry is equal to 46·72 English bushels, or 5 quarters 6 bushels 2·88 pecks.

The following tables of ancient measures have been added from Arbuthnot.

ROMAN MEASURES OF LENGTH.

	Eng. inches.
Digitus transversus	0·72525
Uncia, the ounce	0·967
Palmus minor	2·901
Pes, the foot	11·604
	Eng. feet.
Palmipes	1·20875
Cubitus	1·4505
Gradus	2·4175
	Paces.
Passus	0·967
Stadium	120·875
Milliare	967·0

SCRIPTURE MEASURES OF LENGTH.

	Inches.
Digit	0·7425
Palm	2·97
Span	8·91
	Eng. feet.
Lesser cubit	1·485
Greater cubit	1·7325
	Yards.
Fathom	2·31
Ezekiel's reed	3·465
Arabian pole	4·62
Schænus	46·2
Stadium	231·0
Sabbath day's journey	1155·0
	Miles.
Eastern mile	1·886
Parasang	4·158
Day's journey	33·264

GRECIAN MEASURES OF LENGTH.

	Inches.
Dactylos	0·75546
Doron / Dochme	3·02187
Dichas	7·55468
Orthodoron	8·31015
	Eng. inches.
Spithame	9·06562
Pous	12·0875
	Eng. feet.
Pous	1·00729
Pygme *	1·13203
Pygon	1·25911
Pechys	1·51093
	Eng. paces.
Orgya	1·00729
Stadios / Dulos	100·72916
Milion	805·8333

ROMAN DRY MEASURES.

	Eng. pints.
Hemina	·0·5074
Sextarius	1·0148
	Eng. peck.
Modius	1·0141

ATTIC DRY MEASURES.

	Eng. pints.
Xestes	0·9903
Chenix	1·486
	Eng. bushel.
Medimnus	1·0906

* From this measure is derived the English word pigmy.

JEWISH DRY MEASURES, ACCORDING TO JOSEPHUS.

	Eng. pints.
Gachal	0·1949
Cab	3·874
Gomer	7·0152
	Eng. peck.
Seah	1·4615
	Eng. bush.
Ephah	1·0961
Latech	5·4807
	Eng. quarter.
Coron / Chomer	1·3702

ROMAN MEASURES FOR LIQUIDS.

	Eng. pints.
Hemina	0·59759
Sextarius	1.19518
Congius	7·1712
	Eng. gallons.
Urna	3·5857
Amphora	7·1712
	Eng. hogs.
Culeus	2·2766

ATTIC MEASURES FOR LIQUIDS.

	Eng. pints.
Cotyle	0·5742
Xestes	1·1483
Chous	6·8900
	Eng. gall.
Meteotes	10·3350

JEWISH MEASURES FOR LIQUIDS.

	Eng. pints.
Caph	0·8612
Log	1·1483
Cab	4·5933

	Eng. gall.
Hin	1·7225
Seah	3·4450
Bath	10·3350
	Eng. hogs.
Coron	1·64 05

FRENCH MEASURES.

The English foot is to the Paris foot, as 1 to 1·065977
The English square ft. is to the Paris, as 1 to 1·136307
The English cube ft. is to the Paris, as 1 to 1·211277

The French wine pint contains 58·1413 English cubical inches; and the English wine pint contains 28·875 cubical inches.

NEW FRENCH MEASURES.

The new French measures were established by a decree of the national convention, on the 7th of April 1795. The elementary measure on which they are founded, is a decimal part of the distance from the pole to the equator; that is, a decimal part of a quarter of the terrestrial meridian: for the Metre, which is the element of all the rest, is the ten millionth part of that distance, and is equal, in the old French measures, to 36 inches and 11·296 lines. A metre, in length, is the element of all the lineal measures; a square metre is the element of all the superficial measures; and a cubic metre is the element of all the measures of capacity.

MEASURES OF LENGTH.

	Eng. inches.
Millimetre	·03937
Centimetre	·39378
Decimetre	3·93786
Metre	39·37860
Decametre	393·78605
Hectometre	3937·86059
Chiliometre . . .	39378·60599
Myriometre . . .	393786·05997

	Miles	fur.	yards	feet	inch.
A metre is . .	—	—	—	3	3·37
A Decametre . .	—	—	10	2	9·78
A Hectometre . .	—	—	109	1	1·86
A Chiliometre . .	—	4	213	2	6·60
A Myriometre . .	6	1	158	1	6·05

The distance from the pole to the equator, or 4th part of the terrestrial meridian, according to the late French measurement, is 32815504 English feet.

Centesimal degree = 328155·04 English feet.

MEASURES OF CAPACITY.

	Eng. cub. inches.
Millilitre	·06105
Centilitre	·61063
Decilitre	6·10634
Litre	61·06345
Decalitre	610·63450
Hectolitre	6106·34504
Chiliolitre (cubic metre) .	61063·45042
Myriolitre . . .	610634·50427

A litre is 2·114, or nearly $2\frac{1}{9}$ English wine pints.

A Hectolitre is 2·6434 wine gallons, or 2 gall. 2 quarts 1·14 pint.

A chiliolitre is 4 hogsheads, 12 gallons, 1·36 quart; or 1 tun 12·34 gallons.

A Myriolitre is 10 tuns, 1 hogshead, 60·4 gallons; or nearly $10\frac{1}{2}$ tuns.

SQUARE OR SUPERFICIAL MEASURES.

	Eng. square feet.
Square millimetre . . .	·01076
Square centimetre . . .	·10768
Square decimetre . . .	1·07685
Centiare (square metre) . .	10·76856
Deciare	107·68564
Are	1076·85645
Decare	10768·56454

	Eng. square feet.
Hectare	107685·64540
Chiliare	1076856·45407
Myriare	10768564·54070

A Hectare is 2·472 English statute acres; or 2 acres 1 rood 35·5 poles.

MEASURES FOR FIRE WOOD.

	Eng. cubic feet.
Decistere	3·533764
Stere (cubic metre) . .	35·337645

END OF THE FIRST VOLUME.

T. DAVISON, Lombard-street,
Whitefriars, London.

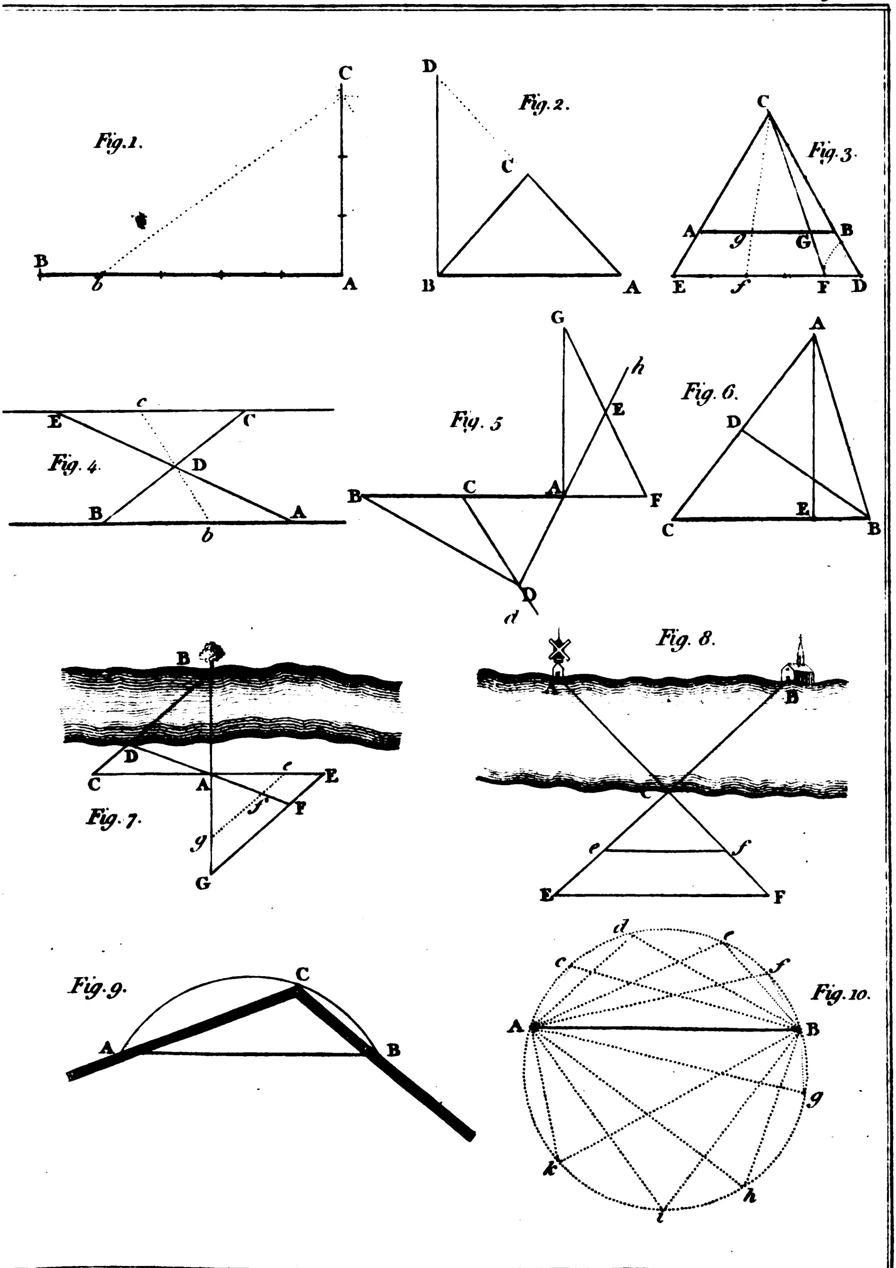


Fig. 1.
C B b A
Fig. 2.
D C B A
Fig. 3.
C A g G B E f F D
Fig. 4.
c E C D B b A
Fig. 5.
G h E B C A F D d
Fig. 6.
A D E C B
Fig. 7.
B D C A e E f F g G
Fig. 8.
A B C e f E F
Fig. 9.
C A B
Fig. 10.
d e c f A B g k h i

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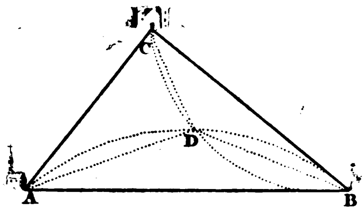
Fig. 11.



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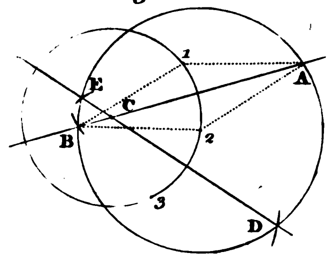

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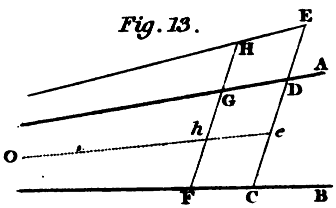

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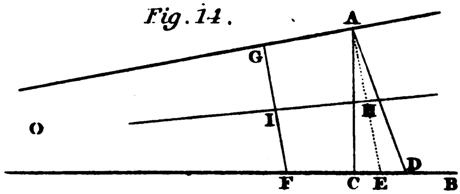

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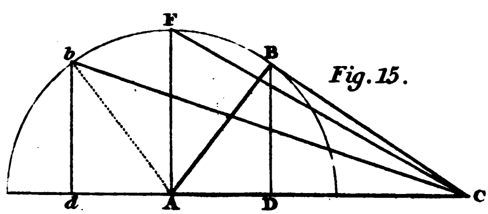

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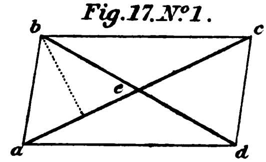

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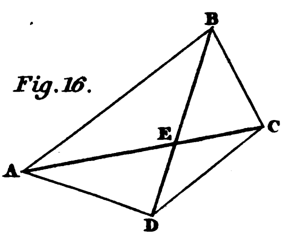

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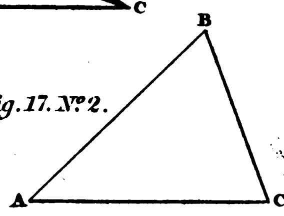

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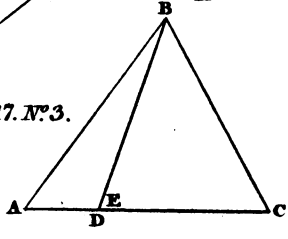

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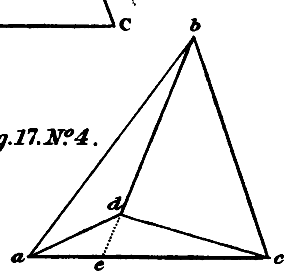

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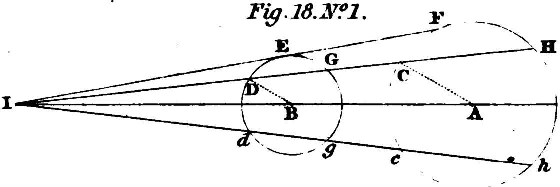
Fig. 18. No. 1.



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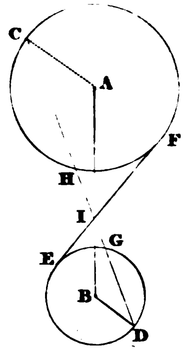

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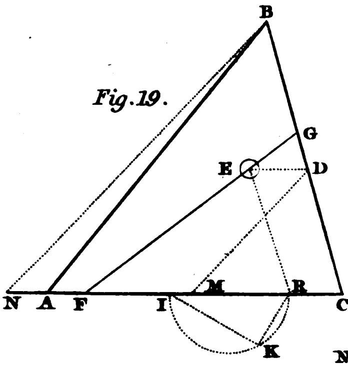

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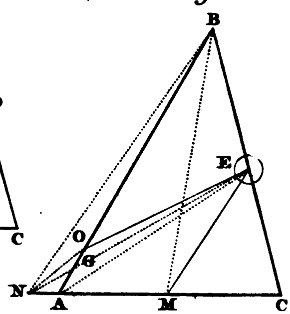

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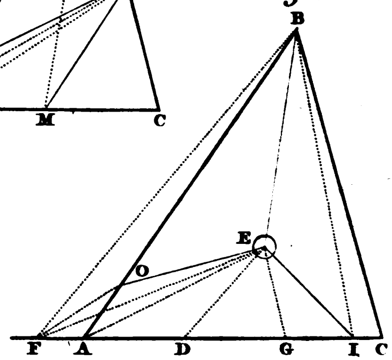

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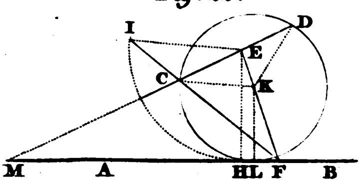

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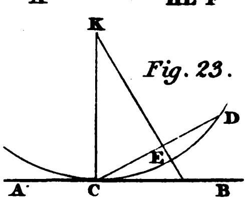

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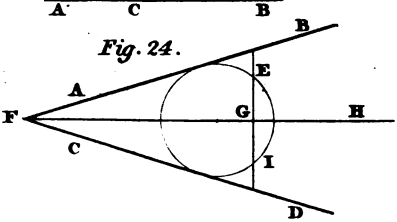

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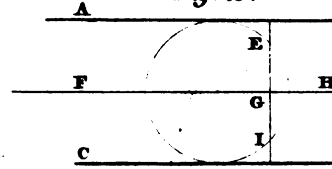

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Fig. 26.

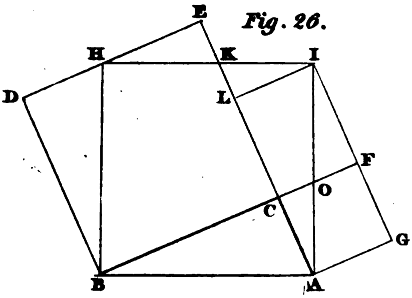


Fig. 27.

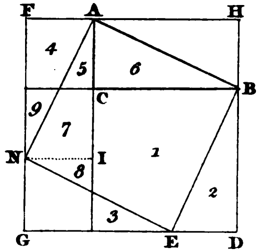
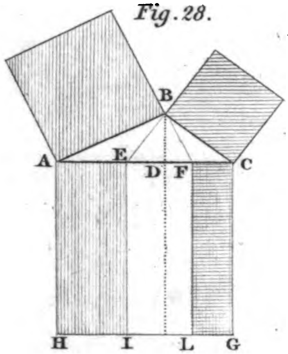


Fig. 28.



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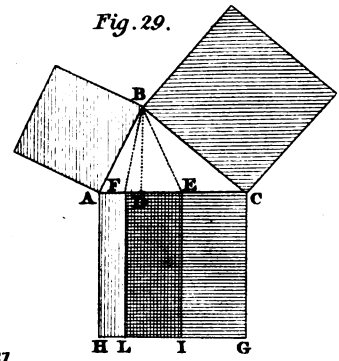

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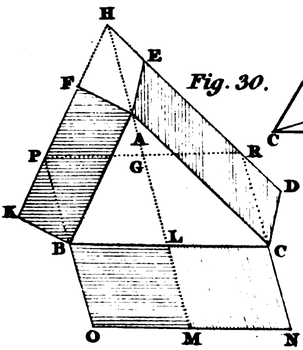

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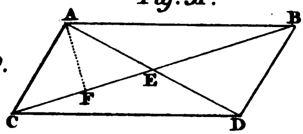

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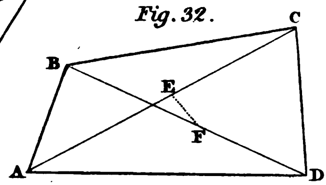

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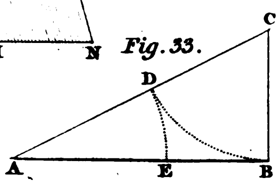

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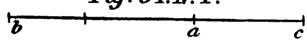

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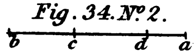

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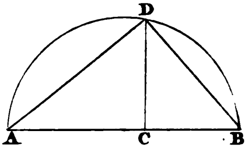

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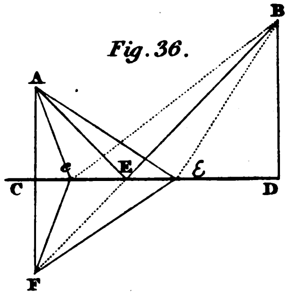

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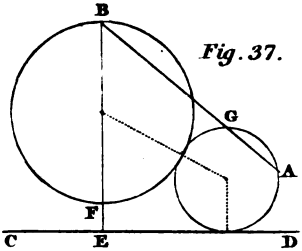

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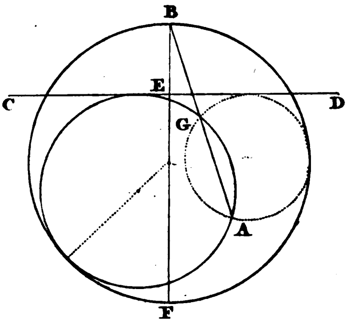

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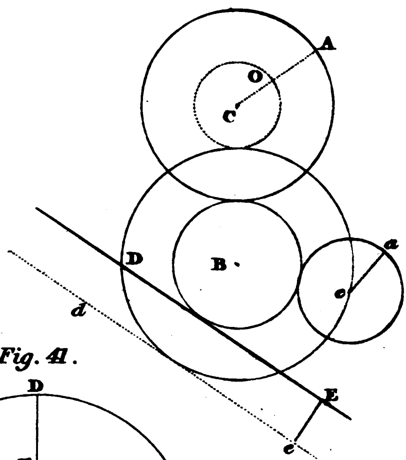

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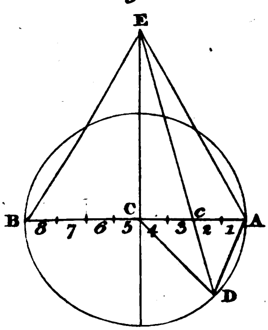

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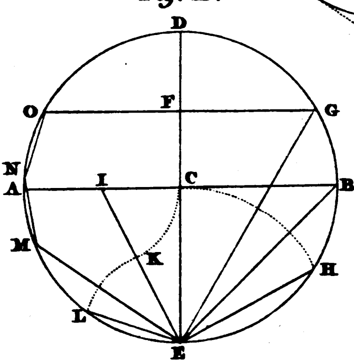

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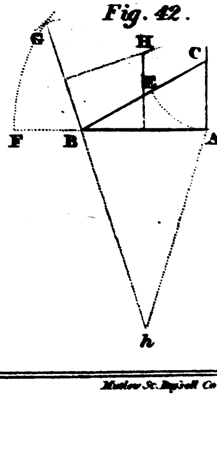

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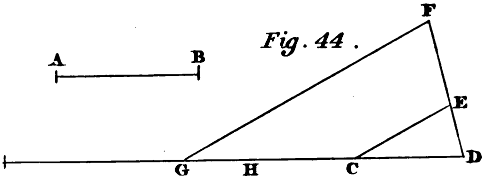

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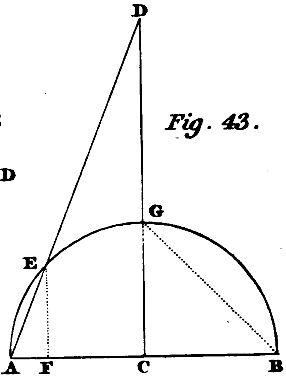

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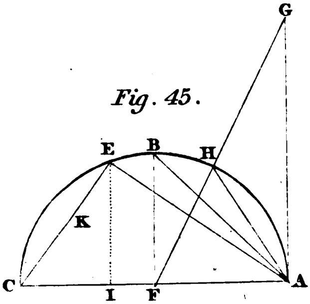
Fig. 45.



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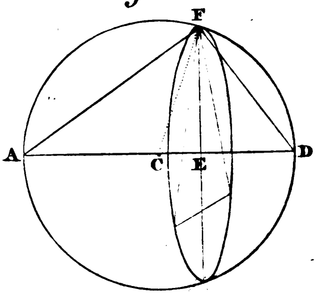

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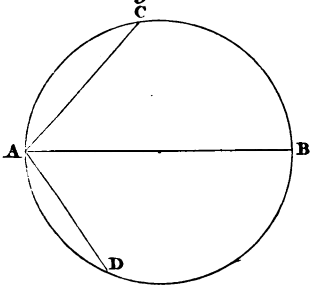

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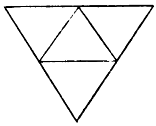

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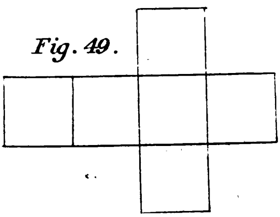

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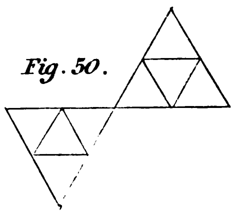

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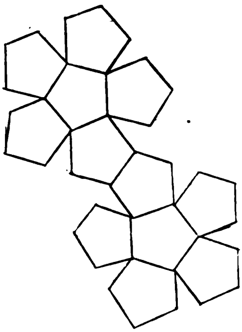

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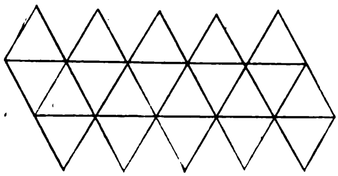

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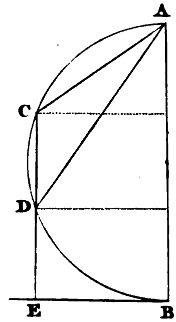

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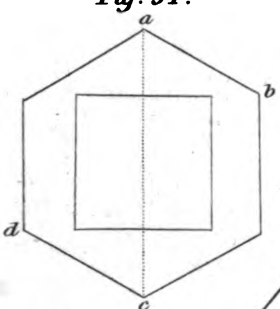

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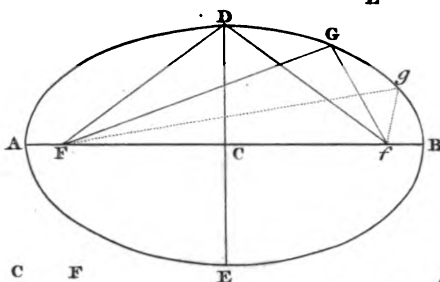

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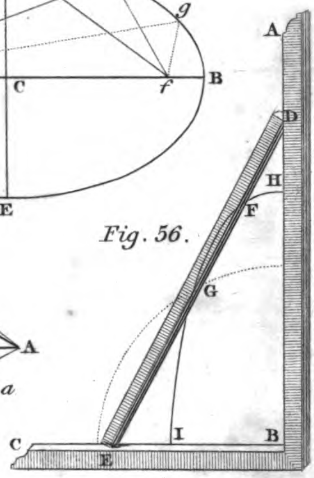

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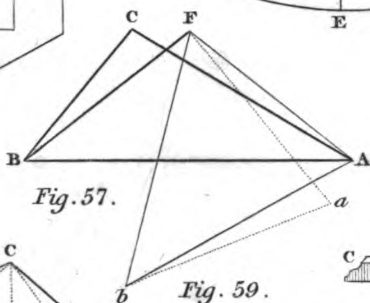

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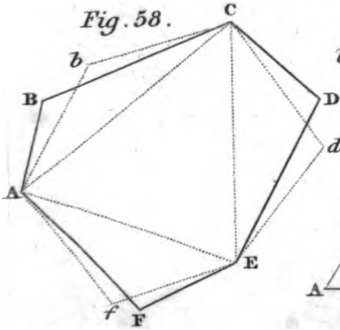

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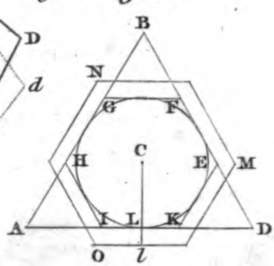

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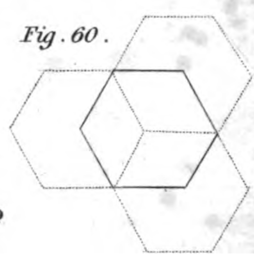

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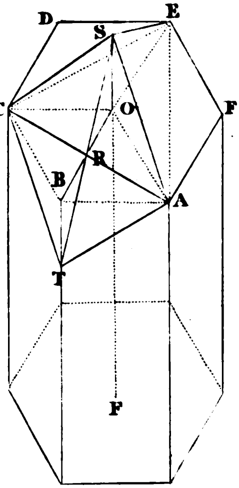

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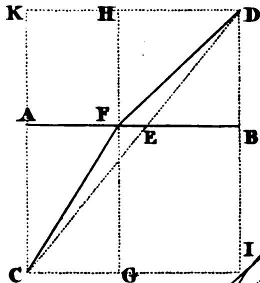

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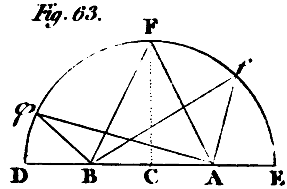

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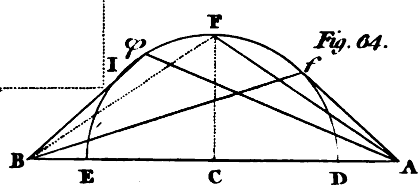

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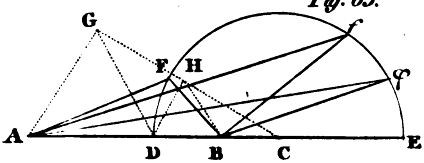

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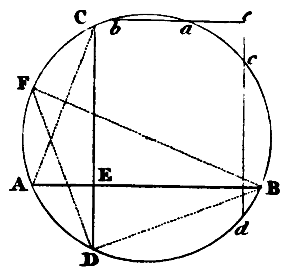

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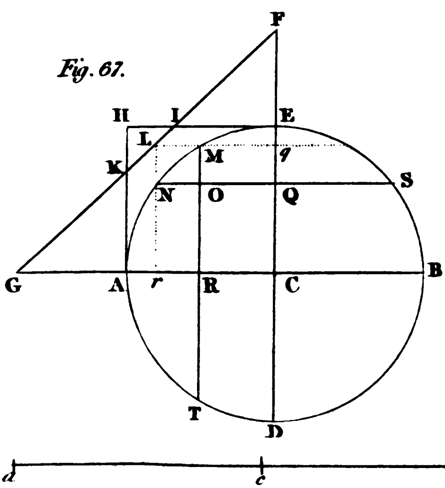

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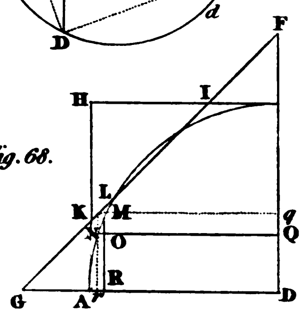

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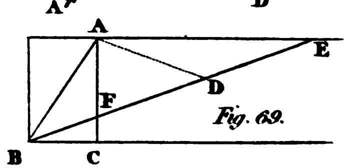

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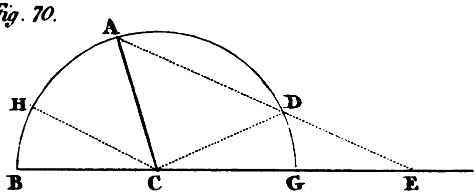

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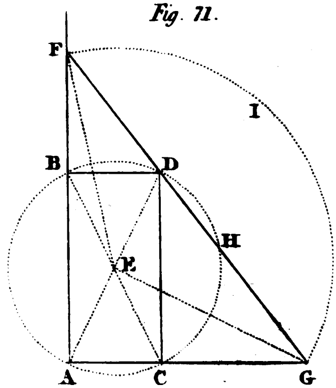

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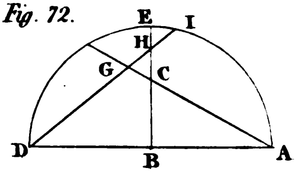

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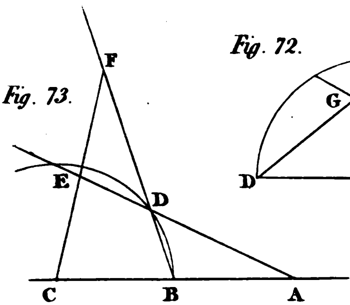

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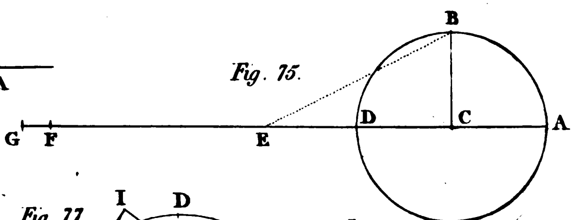

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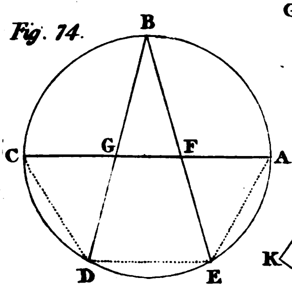

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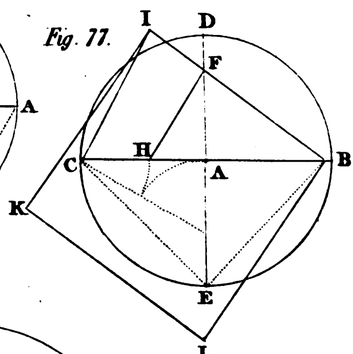

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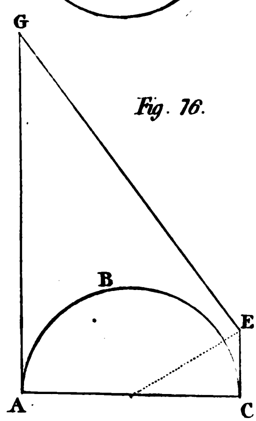

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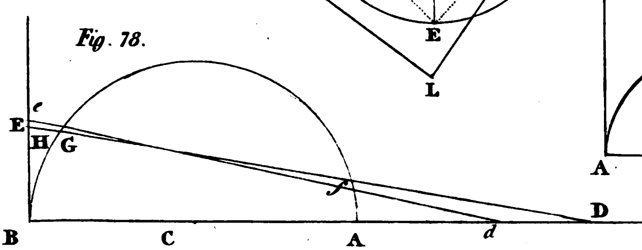

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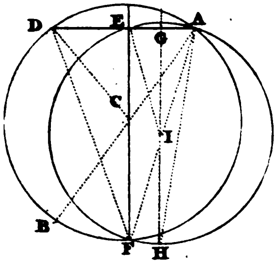

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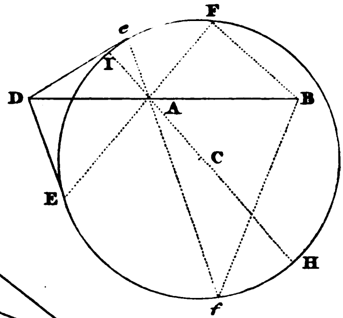

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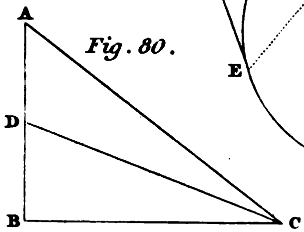

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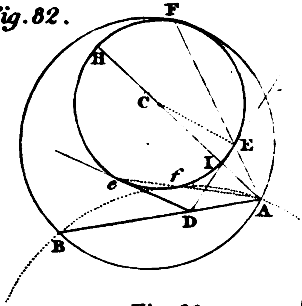

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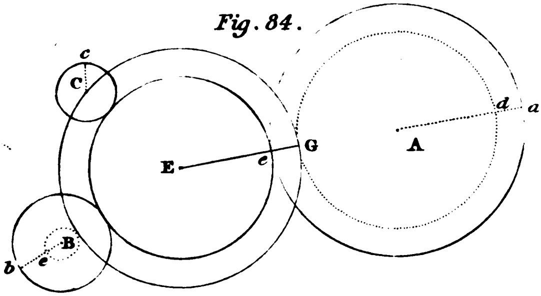

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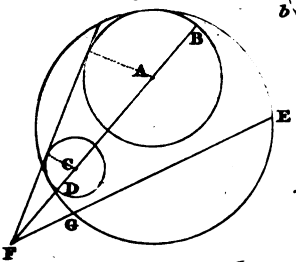

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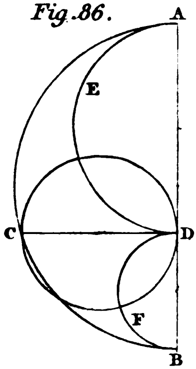

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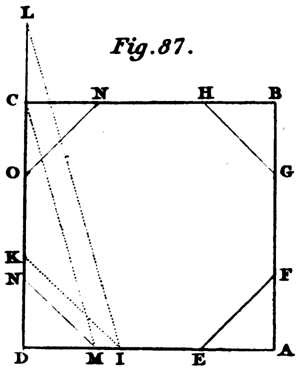

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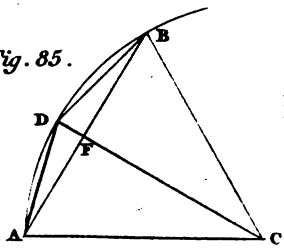

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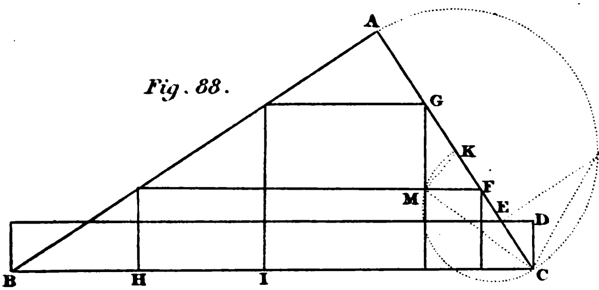

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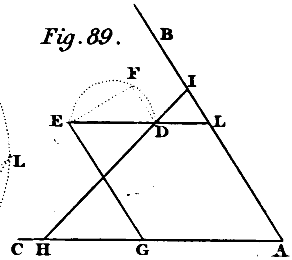

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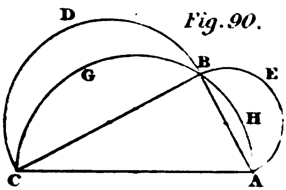

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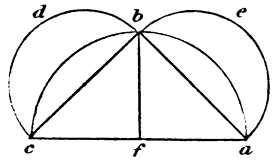

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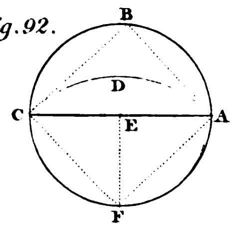

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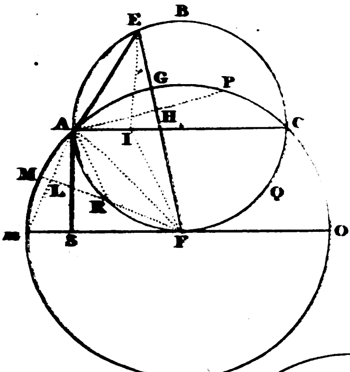

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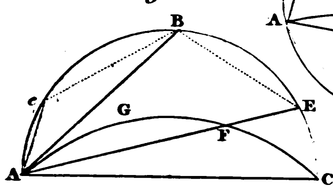

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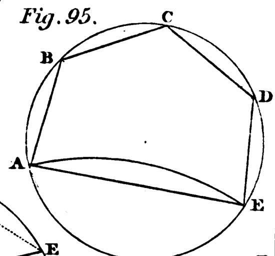

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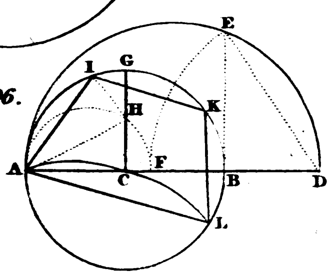

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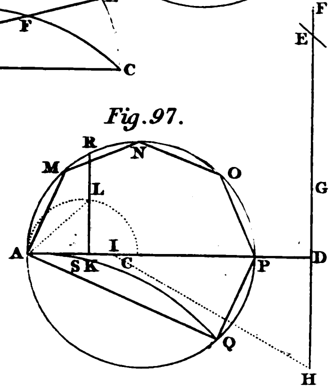

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Fig. 98.

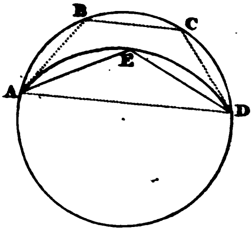


Fig. 99.

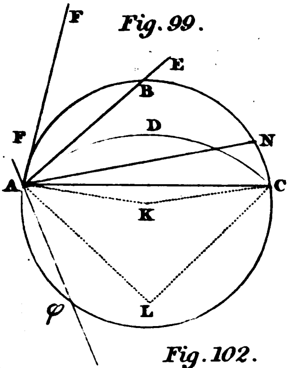
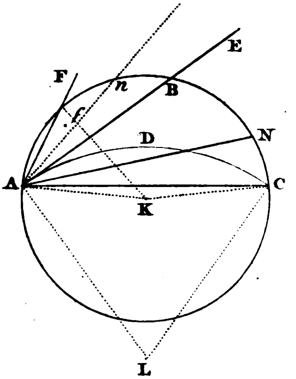


Fig. 100.



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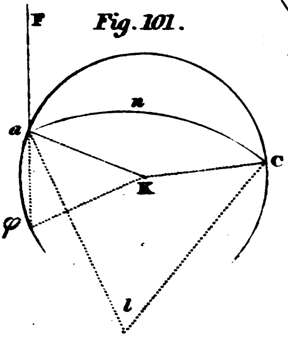

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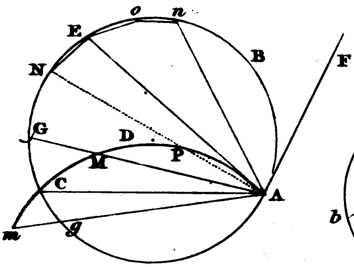

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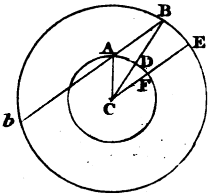

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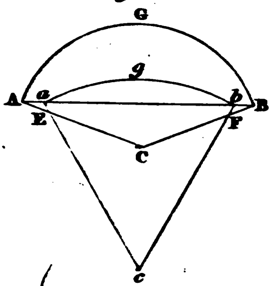

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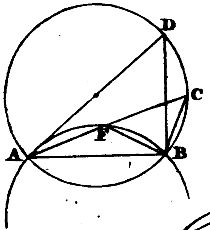

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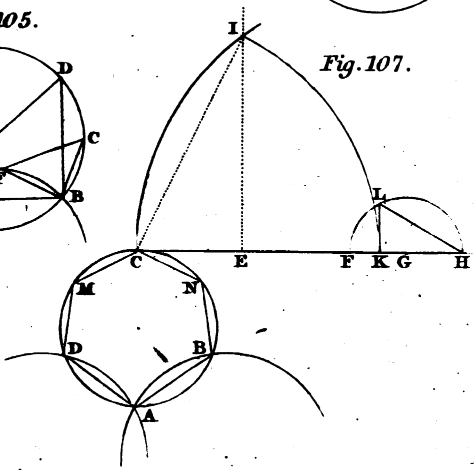

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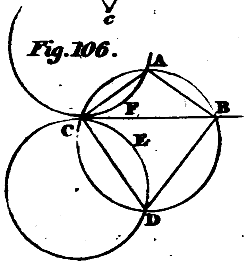
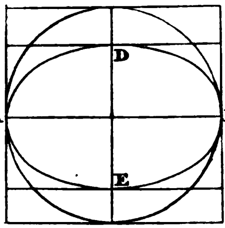


Mutlow Sc. Russell Co.

Fig. 109.



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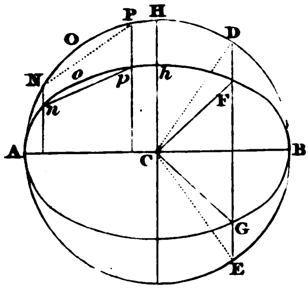

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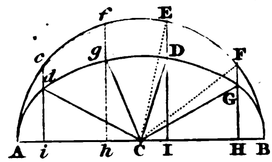

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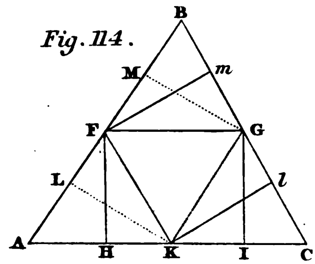

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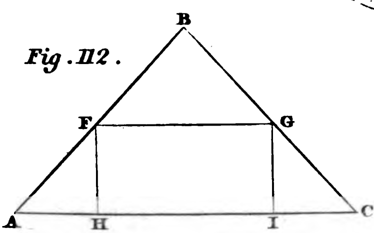

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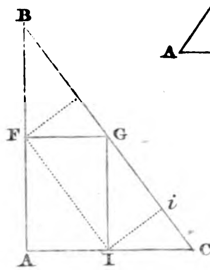

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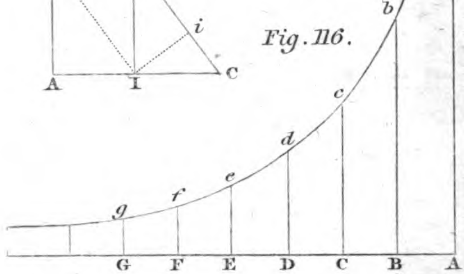

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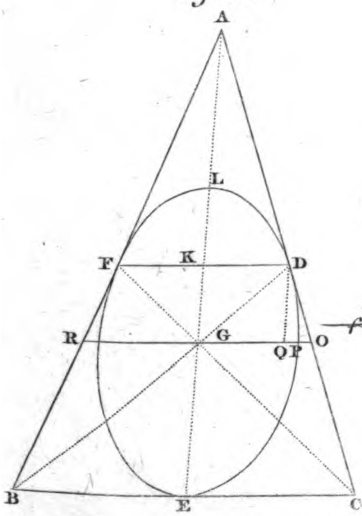


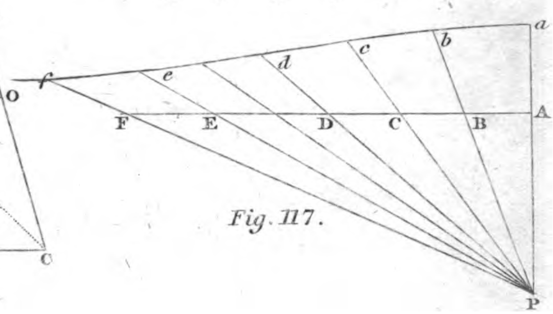
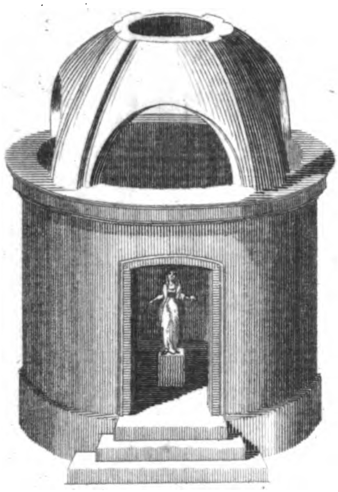


Fig. 117.

Mutlow Sc. Russell Ct.

Fig. 118.



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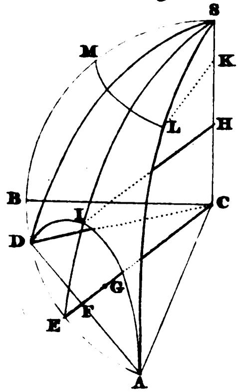

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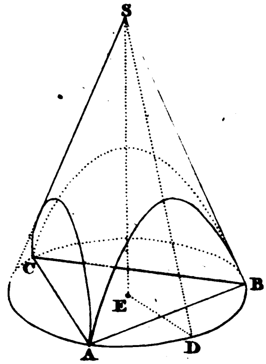

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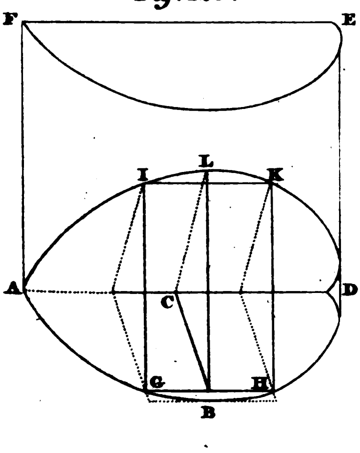
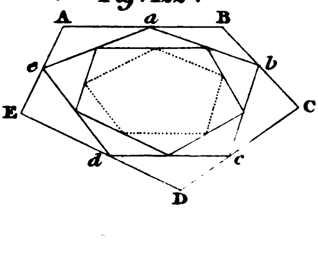
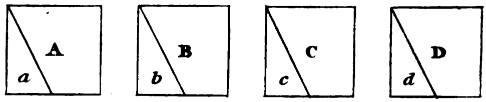


Fig. 122.



Mutlow Sc. Bigbell Co.

Fig. 123. No. 1.



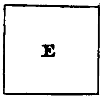


Fig. 123. No. 2.

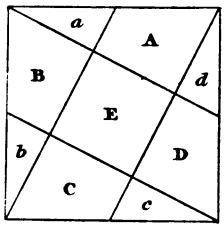
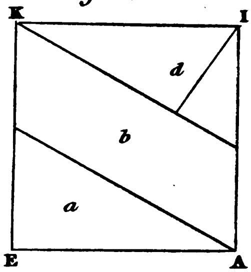


Fig. 125. No. 3.



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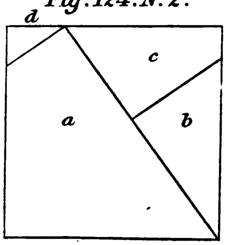

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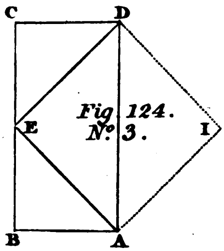

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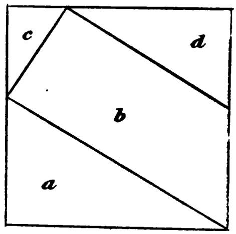

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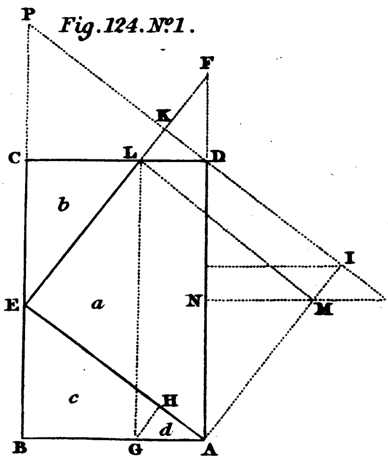
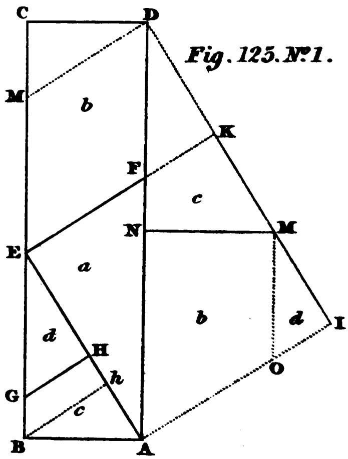
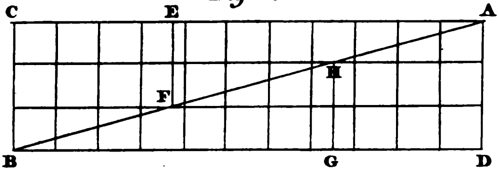


Fig. 125. No. 1.



Mutlow Sc. Russell Co.

Fig. 127. No. 1.



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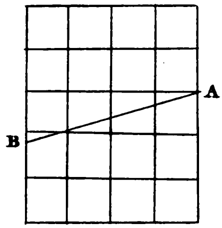

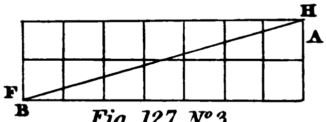

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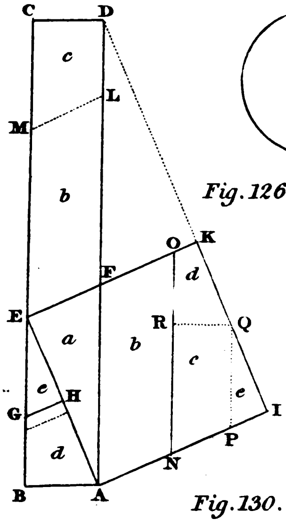
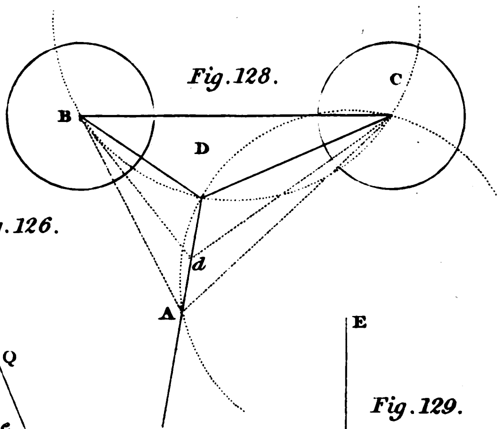


Fig. 126.

Fig. 128.



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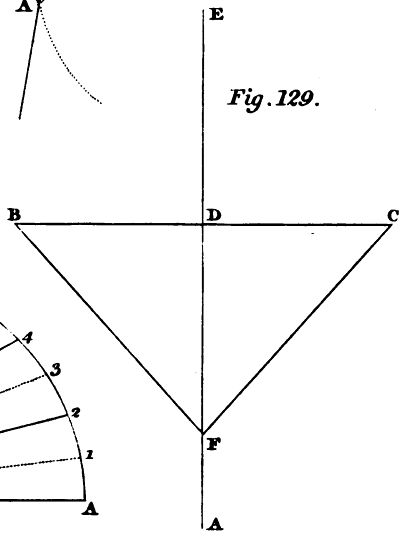
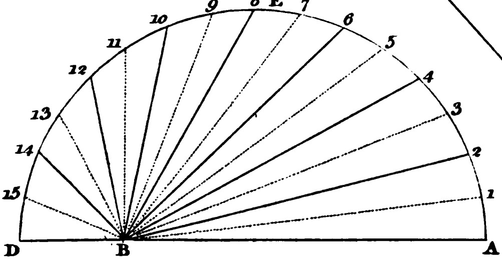
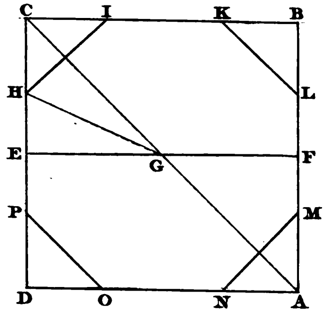


Fig. 130.



Mutlow Sc. Russell Co.

Fig. 131.



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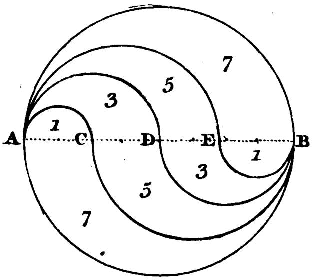
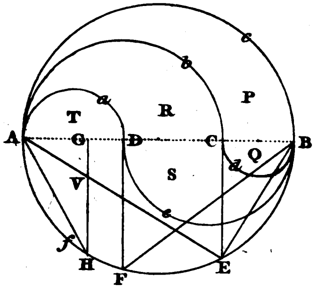


Fig. 133.



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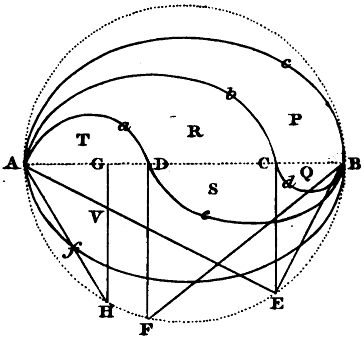

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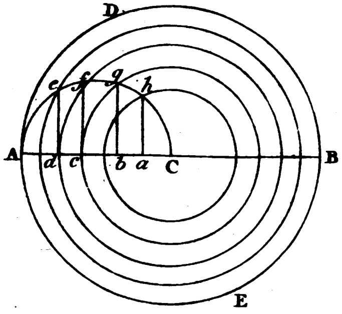

Mutlow Sc. Russell Co.